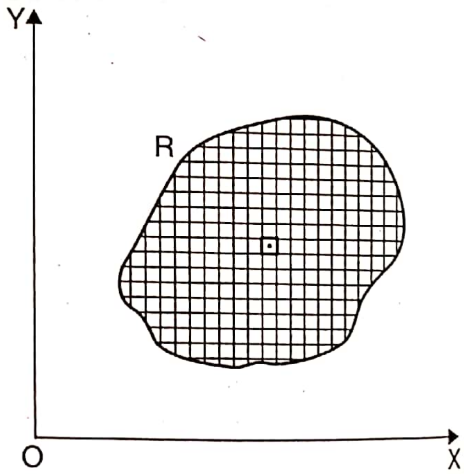
6

Multiple Integrals

6.1. DOUBLE INTEGRALS

The definite integral $\int_a^b f(x)\,dx$ is defined as the limits of the sum $f(x_1)\delta x_1 + f(x_2)\delta x_2 + \cdots + f(x_n)\delta x_n$ when $n \to \infty$ and each of the lengths $\delta x_1, \delta x_2, \cdots, \delta x_n$ tends to zero. Here $\delta x_1, \delta x_2, \cdots, \delta x_n$ are n sub-intervals into which the range $b - a$ has been divided and $x_1, x_2, \cdots, x_n$ are values of x lying respectively in the first, second, $\cdots$, nth sub-interval.

A double integral is its counterpart in two dimensions. Let a single-valued and bounded function $f(x, y)$ of two independent variables x, y be defined in a closed region R of the xy-plane. Divide the region R into sub-regions by drawing lines parallel to co-ordinate axes. Number the rectangles which lie entirely inside the region R, from 1 to n. Let (x_r, y_r) be any point inside the r^{th} rectangle whose area is δA_r.



Consider the sum

$$f(x_1, y_1)\delta A_1 + f(x_2, y_2)\delta A_2 + \cdots + f(x_n, y_n)\delta A_n$$

$$= \sum_{r=1}^{n} f(x_r, y_r)\delta A_r \qquad \ldots(1)$$

Let the number of these sub-regions increase indefinitely, such that the largest linear dimension (*i.e.*, diagonal) of δA_r approaches zero. The limit of the sum (1), if it exists, irrespective of the mode of sub-division, is called the *double integral* of $f(x, y)$ over the region R and is denoted by $\iint_R f(x, y)\,dA$

In other words,

$$\underset{\substack{n \to \infty \\ \delta A_r \to 0}}{\text{Lt}} \sum_{r=1}^{n} f(x_r, y_r)\,\delta A_r = \iint_R f(x, y)\,dA$$

which is also expressed as

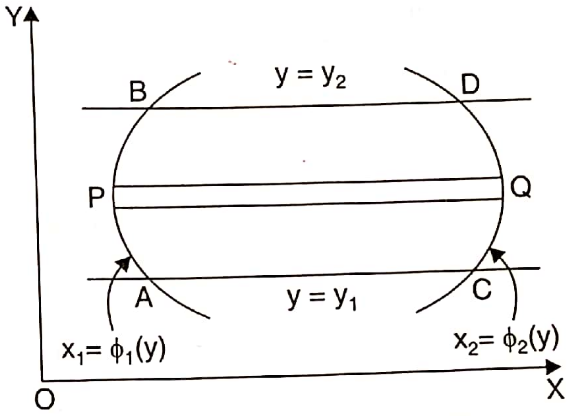
$$\iint_R f(x, y)\,dx\,dy \quad \text{or} \quad \iint_R f(x, y)\,dy\,dx$$

6.2. EVALUATION OF DOUBLE INTEGRALS

The methods of evaluating the double integrals depend upon the nature of the curves bounding the region R. Let the region R be bounded by the curves $x = x_1$, $x = x_2$ and $y = y_1$, $y = y_2$.

(*i*) **When x_1, x_2 are functions of y and y_1, y_2 are constants.** Let AB and CD be the curves $(x_1 = \phi_1(y))$ and $x_2 = \phi_2(y)$.

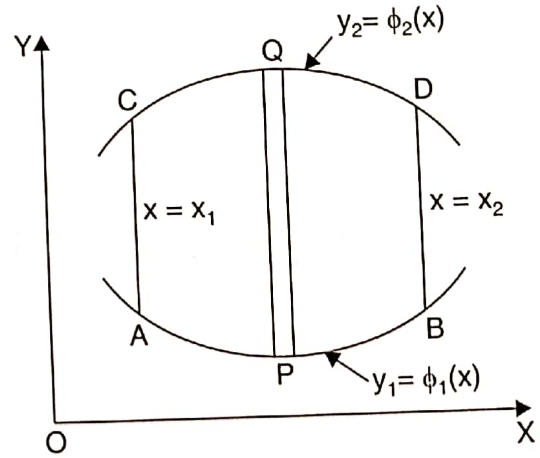
Take a horizontal strip PQ of width δy. Here the double integral is evaluated first w.r.t. x (treating y as a constant). The resulting expression which is a function of y is integrated w.r.t. y between the limits $y = y_1$ and $y = y_2$. Thus



$$\iint_R f(x, y)\, dx\, dy = \boxed{\int_{y_1}^{y_2} \boxed{\int_{x_1=\phi_1(y)}^{x_2=\phi_2(y)} f(x,y)\, dx}\; dy}$$

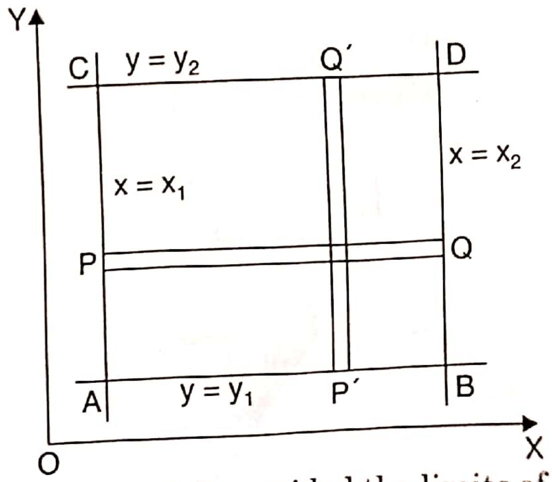
the integration being carried from the inner to the outer rectangle. Geometrically, the integral in the inner rectangle indicates that the integration is performed along the horizontal strip PQ (keeping y constant) while the outer rectangle corresponds to the sliding of the strip PQ from AC to BD thus covering the entire region ABDC of integration.

(*ii*) **When y_1, y_2 are functions of x and x_1, x_2 are constants.** Let AB and CD be the curves $(y_1 = \phi_1(x))$ and $y_2 = \phi_2(x)$. Take a vertical strip PQ of width δx. Here the double integral is evaluated first w.r.t. y (treating x as constant). The resulting expression which is a function of x is integrated w.r.t. x between the limits $x = x_1$ and $x = x_2$. Thus.



$$\iint_R f(x, y)\, dx\, dy = \boxed{\int_{x_1}^{x_2} \boxed{\int_{y_1=\phi_1(x)}^{y_2=\phi_2(x)} f(x, y)\, dy}\; dx}$$

the integration being carried from the inner to the outer rectangle. Geometrically, the integral in the inner rectangle indicates that the integration is performed along the vertical strip PQ (keeping x constant) while the outer rectangle corresponds to the sliding of the strip PQ from AC to BD thus covering the entire region ABDC of integration.



(*iii*) **When x_1, x_2, y_1, y_2 are constants.** Here the region of integration R is the rectangle ABDC. It is immaterial whether we integrate first along the horizontal strip PQ and then slide it from AB to CD; or we integrate first along the vertical strip P′Q′ and then slide it from AC to BD. Thus the order of integration is immaterial, provided the limits of integration are changed accordingly.

$$\iint_R f(x, y)\, dx\, dy = \boxed{\int_{y_1}^{y_2} \boxed{\int_{x_1}^{x_2} f(x, y)\, dx}\, dy} = \boxed{\int_{x_1}^{x_2} \boxed{\int_{y_1}^{y_2} f(x, y)\, dy}\, dx}$$

Note 1. From cases (*i*) and (*ii*) above, we observe that *integration is to be performed w.r.t. that variable having variable limits first and then w.r.t. the variable with constant limits.*

Note 2. If $f(x, y)$ has discontinuities within or on the boundary of the region of integration, then the change of the order of integration does not result into the same integrals.

ILLUSTRATIVE EXAMPLES

Example 1. *Prove that* $\int_1^2 \int_3^4 (xy + e^y)\, dy\, dx = \int_3^4 \int_1^2 (xy + e^y)\, dx\, dy.$

Sol. $\int_1^2 \int_3^4 (xy + e^y)\, dy\, dx = \int_1^2 \left[\int_3^4 (xy + e^y)\, dy\right] dx$

$$= \int_1^2 \left[\frac{xy^2}{2} + e^y\right]_3^4 dx = \int_1^2 \left(8x + e^4 - \frac{9}{2}x - e^3\right) dx$$

$$= \int_1^2 \left(\frac{7}{2}x + e^4 - e^3\right) dx = \left[\frac{7x^2}{4} + (e^4 - e^3)x\right]_1^2$$

$$= 7 + 2(e^4 - e^3) - \frac{7}{4} - (e^4 - e^3) = \frac{21}{4} + e^4 - e^3$$

$$\int_3^4 \int_1^2 (xy + e^y)\, dx\, dy = \int_3^4 \left[\int_1^2 (xy + e^y)\, dx\right] dy = \int_3^4 \left[\frac{yx^2}{2} + xe^y\right]_1^2 dy$$

$$= \int_3^4 \left(2y + 2e^y - \frac{y}{2} - e^y\right) dy = \int_3^4 \left(\frac{3y}{2} + e^y\right) dy$$

$$= \left[\frac{3y^2}{4} + e^y\right]_3^4 = 12 + e^4 - \frac{27}{4} - e^3 = \frac{21}{4} + e^4 - e^3$$

Hence the result.

Example 2. *Evaluate the following:*

(*i*) $\int_0^1 \int_0^1 \frac{dx\, dy}{\sqrt{(1-x^2)(1-y^2)}}$ (*ii*) $\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2)\, dy\, dx.$ **(P.T.U., Dec. 2013)**

Sol. (*i*) $\int_0^1 \int_0^1 \frac{dx\, dy}{\sqrt{(1-x^2)(1-y^2)}} = \int_0^1 \left[\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-y^2)}}\right] dy = \int_0^1 \frac{1}{\sqrt{1-y^2}} \left[\sin^{-1} x\right]_0^1 dy$

$$= \int_0^1 \frac{1}{\sqrt{1-y^2}} \cdot \frac{\pi}{2}\, dy = \frac{\pi}{2}\left[\sin^{-1} y\right]_0^1 = \frac{\pi}{2} \cdot \frac{\pi}{2} = \frac{\pi^2}{4}.$$

(ii) $\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2)\, dy\, dx = \int_0^1 \left[x^2 y + \frac{y^3}{3} \right]_x^{\sqrt{x}} dx$

$$= \int_0^1 \left[x^{5/2} + \frac{x^{3/2}}{3} - \left(x^3 + \frac{x^3}{3} \right) \right] dx$$

$$= \left[\frac{2}{7} x^{7/2} + \frac{1}{3} \cdot \frac{2}{5} x^{5/2} - \frac{x^4}{4} - \frac{x^4}{12} \right]_0^1 = \frac{2}{7} + \frac{2}{15} - \frac{1}{4} - \frac{1}{12} = \frac{3}{35}.$$

Example 3. *Show that* $\int_0^1 dx \int_0^1 \frac{x - y}{(x + y)^3}\, dy \neq \int_0^1 dy \int_0^1 \frac{x - y}{(x + y)^3}\, dx.$

Sol. LHS $= \int_0^1 dx \int_0^1 \frac{2x - (x + y)}{(x + y)^3}\, dy = \int_0^1 dx \int_0^1 \left[\frac{2x}{(x + y)^3} - \frac{1}{(x + y)^2} \right] dy$

$$= \int_0^1 \left[2x \cdot \frac{(x + y)^{-2}}{-2} - \frac{(x + y)^{-1}}{-1} \right]_0^1 dx = \int_0^1 \left[\frac{-x}{(x + y)^2} + \frac{1}{x + y} \right]_0^1 dx$$

$$= \int_0^1 \left[\frac{-x}{(x + 1)^2} + \frac{1}{x + 1} + \frac{1}{x} - \frac{1}{x} \right] dx = \int_0^1 \frac{-x + x + 1}{(x + 1)^2}\, dx = \int_0^1 \frac{1}{(x + 1)^2}\, dx$$

$$= \left[-\frac{1}{x + 1} \right]_0^1 = -\frac{1}{2} + 1 = \frac{1}{2}$$

RHS $= \int_0^1 dy \int_0^1 \frac{(x + y) - 2y}{(x + y)^3}\, dx = \int_0^1 dy \int_0^1 \left[\frac{1}{(x + y)^2} - \frac{2y}{(x + y)^3} \right] dx$

$$= \int_0^1 \left[\frac{(x + y)^{-1}}{-1} - 2y \cdot \frac{(x + y)^{-2}}{-2} \right]_0^1 dy = \int_0^1 \left[-\frac{1}{x + y} + \frac{y}{(x + y)^2} \right]_0^1 dy$$

$$= \int_0^1 \left[-\frac{1}{1 + y} + \frac{y}{(1 + y)^2} + \frac{1}{y} - \frac{1}{y} \right] dy = \int_0^1 \frac{-1 - y + y}{(1 + y)^2}\, dy = -\int_0^1 \frac{1}{(1 + y)^2}\, dy$$

$$= \left[\frac{1}{1 + y} \right]_0^1 = \frac{1}{2} - 1 = -\frac{1}{2}$$

$\therefore$ The two integrals are not equal.

Example 4. *Evaluate* $\int_0^1 \int_0^{\sqrt{1 + x^2}} \frac{dy\, dx}{1 + x^2 + y^2}$. **(P.T.U., May 2006)**

Sol. $\quad \mathrm{I} = \int_0^1 \left[\int_0^{\sqrt{1 + x^2}} \frac{1}{(1 + x^2) + y^2}\, dy \right] dx$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \left[\tan^{-1} \frac{y}{\sqrt{1+x^2}} \right]_0^{\sqrt{1+x^2}} dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} [\tan^{-1} 1 - \tan^{-1} 0] \, dx = \frac{\pi}{4} \int_0^1 \frac{dx}{\sqrt{1+x^2}}$$

$$= \frac{\pi}{4} \left[\log (x + \sqrt{1+x^2}) \right]_0^1 = \frac{\pi}{4} [\log (1 + \sqrt{2}) - \log 1] = \frac{\pi}{4} \log (\sqrt{2} + 1).$$

Example 5. *Evaluate* $\int_0^1 \int_{x^2}^{2-x} xy \, dy \, dx$ **(P.T.U., Dec. 2012)**

Sol. $\int_0^1 \int_{x^2}^{2-x} xy \, dy \, dx = \int_0^1 x \frac{y^2}{2} \Big|_{x^2}^{2-x} dx = \int_0^1 \frac{x}{2} \{(2-x)^2 - x^4\} \, dx$

$$= \int_0^1 \frac{x}{2} \{4 - 4x + x^2 - x^4\} \, dx = \frac{1}{2} \int_0^1 (4x - 4x^2 + x^3 - x^5) \, dx$$

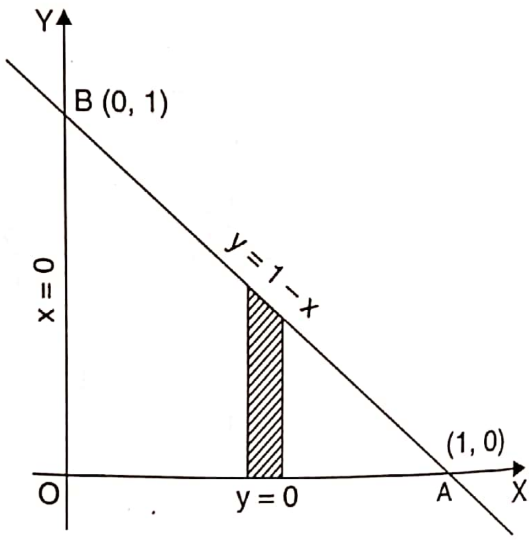
$$= \frac{1}{2} \left\{ 2x^2 - \frac{4x^3}{3} + \frac{x^4}{4} - \frac{x^6}{6} \right\} \Big|_0^1 = \frac{1}{2} \left(2 - \frac{4}{3} + \frac{1}{4} - \frac{1}{6} \right)$$

$$= \frac{1}{2} \cdot \frac{9}{12} = \frac{9}{24}$$

Example 6. (*i*) *Evaluate* $\iint e^{2x + 3y} \, dx \, dy$ *over the triangle bounded by* $x = 0$, $y = 0$ *and* $x + y = 1$. ⇒ y = 1 - x , x = 1

(*ii*) *Evaluate* $\int \int_S \sqrt{xy - y^2} \, dxdy$ *where S is the triangle with vertices (0, 0), (10, 1) and (1, 1).*

Sol. (*i*) The region R of integration is the triangle OAB. Here x varies from 0 to 1 and y varies from x-axis upto the line $x + y = 1$ *i.e.*, from 0 to $1 - x$.



∴ The region R can be expressed as

$$0 \le x \le 1, \; 0 \le y \le 1 - x$$

$$\therefore \iint_R e^{2x+3y} \, dx \, dy = \int_0^1 \int_0^{1-x} e^{2x+3y} \, dy \, dx$$

$$= \int_0^1 \left[\frac{1}{3} e^{2x+3y} \right]_0^{1-x} dx$$

$$= \frac{1}{3} \int_0^1 (e^{3-x} - e^{2x}) \, dx$$

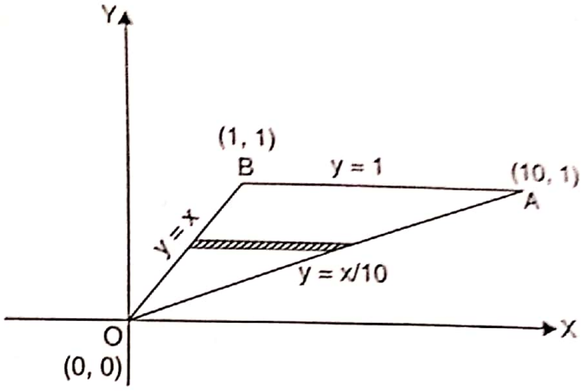
$$= \frac{1}{3} \left[-e^{3-x} - \frac{1}{2} e^{2x} \right]_0^1 = -\frac{1}{3} \left[\left(e^2 + \frac{1}{2} e^2 \right) - \left(e^3 + \frac{1}{2} \right) \right]$$

$$= -\frac{1}{3} \left[-e^2(e-1) + \frac{1}{2}(e^2 - 1) \right]$$

$$= \frac{1}{6}(e-1)\,[2e^2-(e+1)] = \frac{1}{6}(e-1)(2e^2-e-1)$$

$$= \frac{1}{6}(e-1)(e-1)(2e+1) = \frac{1}{6}(e-1)^2(2e+1).$$

(*ii*) Region R of integration is the region of the triangle OAB (shown in the figure). Here equation of the line OA is



$$y - 0 = \frac{1-0}{10-0}(x-0)$$

or $$y = \frac{1}{10}x$$

equation of OB is $$y = x$$

$\therefore$ $\quad$ R = $\{(x, y)\ ;\ y \le x \le 10y\ ;\ 0 \le y \le 1\}$

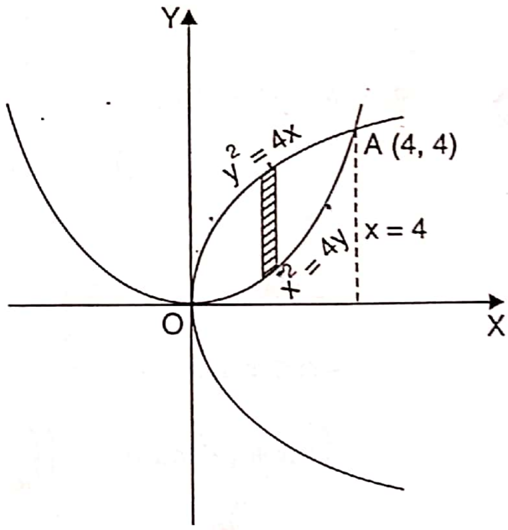
$$\therefore \quad \int\int_S \sqrt{xy-y^2}\,dxdy = \int_0^1 \left[\int_y^{10y} \sqrt{xy-y^2}\,dx\right] dy$$

$$= \int_0^1 \frac{(xy-y^2)^{3/2}}{\frac{3}{2}y}\Bigg|_y^{10y} dy = \frac{2}{3}\int_0^1 \left[\frac{(9y^2)^{3/2}}{y} - 0\right] dy$$

$$= \frac{2}{3}\int_0^1 27y^2\,dy = \frac{2}{3}\left[27\,\frac{y^3}{3}\Bigg|_0^1\right] = \frac{2}{3}\cdot 9 = 6.$$

Example 7. *Evaluate* $\iint_R y\,dx\,dy$, *where R is the region bounded by the parabolas* $y^2 = 4x$ *and* $x^2 = 4y$.

Sol. Solving $y^2 = 4x$ and $x^2 = 4y$, we have



$$\left(\frac{x^2}{4}\right)^2 = 4x \quad \text{or} \quad x(x^3 - 64) = 0$$

$\therefore$ $\quad x = 0, 4$

When $\quad x = 4,\ y = 4$

$\therefore$ Co-ordinates of A are (4, 4)

The region R can be expressed as

$$0 \le x \le 4,\ \frac{x^2}{4} \le y \le 2\sqrt{x}$$

$$\therefore \quad \iint_R y\,dx\,dy = \int_0^4 \int_{x^2/4}^{2\sqrt{x}} y\,dy\,dx$$

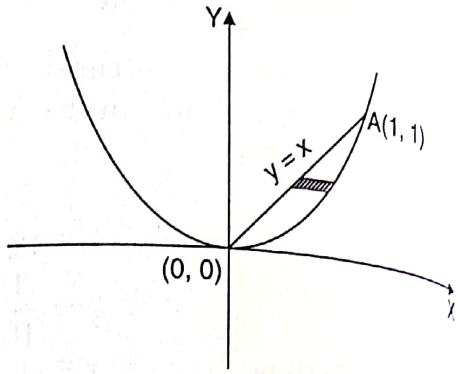
$$= \int_0^4 \frac{1}{2}\left[y^2\right]_{x^2/4}^{2\sqrt{x}} dx = \frac{1}{2}\int_0^4 \left(4x - \frac{x^4}{16}\right) dx$$

$$= \frac{1}{2}\left[2x^2 - \frac{x^5}{80}\right]_0^4 = \frac{1}{2}\left[32 - \frac{1024}{80}\right] = \frac{48}{5}.$$

Example 8. *Evaluate* $\iint xy(x+y)\,dxdy$ *over the area between* $y = x^2$ *and* $y = x$.

Sol. Region of integration is

$$R = \left\{(x, y); y \le x \le \sqrt{y}; 0 \le y \le 1\right\}$$



$$\therefore \iint_R xy\,(x+y)\,dxdy$$

$$= \int_0^1 \left[\int_y^{\sqrt{y}} xy\,(x+y)\,dx\right] dy$$

$$= \int_0^1 y\,\frac{x^3}{3} + y^2\,\frac{x^2}{2}\Bigg|_y^{\sqrt{y}}\,dy$$

$$= \int_0^1 \left(\frac{1}{3}y^{5/2} + \frac{1}{2}y^3 - \frac{y^4}{3} - \frac{y^4}{2}\right) dy$$

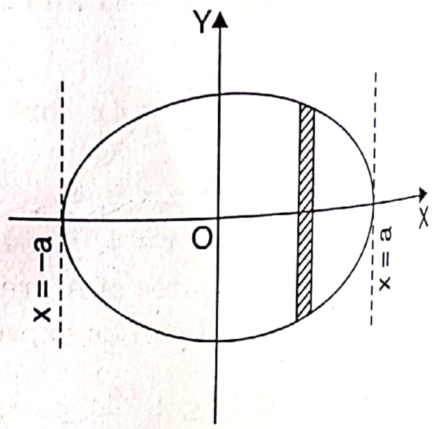
$$= \int_0^1 \left(\frac{1}{3}y^{5/2} + \frac{1}{2}y^3 - \frac{5}{6}y^4\right) dy = \frac{1}{3}\left(\frac{2}{7}y^{7/2}\right) + \frac{1}{2}\cdot\frac{y^4}{4} - \frac{5}{6}\frac{y^5}{5}\Bigg|_0^1$$

$$= \frac{2}{21} + \frac{1}{8} - \frac{1}{6} = \frac{16+21-28}{168} = \frac{9}{168} = \frac{3}{56}.$$

Example 9. *Evaluate* $\iint (x+y)^2\,dx\,dy$ *over the area bounded by the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Sol. For the ellipse



$$\frac{y}{b} = \pm\sqrt{1 - \frac{x^2}{a^2}} \quad \text{or} \quad y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$$

$\therefore$ The region of integration R can be expressed as

$$-a \le x \le a, \; -\frac{b}{a}\sqrt{a^2 - x^2} \le y \le \frac{b}{a}\sqrt{a^2 - x^2}$$

$$\therefore \iint_R (x+y)^2\,dx\,dy = \iint_R (x^2 + y^2 + 2xy)\,dx\,dy$$

$$= \int_{-a}^{a} \int_{-b/a\sqrt{a^2-x^2}}^{b/a\sqrt{a^2-x^2}} (x^2 + y^2 + 2xy)\,dy\,dx$$

$$= \int_{-a}^{a} \int_{-b/a\sqrt{a^2-x^2}}^{b/a\sqrt{a^2-x^2}} (x^2 + y^2)\,dy\,dx + \int_{-a}^{a} \int_{-b/a\sqrt{a^2-x^2}}^{b/a\sqrt{a^2-x^2}} 2xy\,dy\,dx$$

$$= \int_{-a}^{a} \int_{0}^{b/a\sqrt{a^2-x^2}} 2(x^2+y^2)\, dy\, dx + 0$$

[since $(x^2 + y^2)$ is an even function of y and $2xy$ is an odd function of y]

$$= \int_{-a}^{a} \left[2\left(x^2 y + \frac{y^3}{3} \right) \right]_0^{b/a\sqrt{a^2-x^2}} dx$$

$$= 2\int_{-a}^{a} \left[x^2 \cdot \frac{b}{a}\sqrt{a^2-x^2} + \frac{1}{3}\cdot\frac{b^3}{a^3}(a^2-x^2)^{3/2} \right] dx$$

$$= 4\int_{0}^{a} \left[\frac{b}{a}x^2\sqrt{a^2-x^2} + \frac{b^3}{3a^3}(a^2-x^2)^{3/2} \right] dx$$

put $x = a\sin\theta$, $\therefore$ $dx = a\cos\theta\, d\theta$

$$= 4\int_{0}^{\pi/2} \left(\frac{b}{a}\cdot a^2\sin^2\theta \cdot a\cos\theta + \frac{b^3}{3a^3}\cdot a^3\cos^3\theta \right) \times a\cos\theta\, d\theta$$

$$= 4\int_{0}^{\pi/2} \left(a^3 b \sin^2\theta\cos^2\theta + \frac{ab^3}{3}\cos^4\theta \right) d\theta$$

We know from integral calculus that

$$\int_0^{\frac{\pi}{2}} \sin^p x \cos^q x\, dx = \frac{\{(p-1)(p-3)\cdots\}\{(q-1)(q-3)\cdots\}}{(p+q)(p+q-2)\cdots}\frac{\pi}{2}$$

if both p, q are even otherwise no $\frac{\pi}{2}$

and
$$\int_0^{\frac{\pi}{2}} \cos^p x\, dx = \int_0^{\frac{\pi}{2}} \sin^p x\, dx = \frac{(p-1)(p-3)\cdots}{p(p-2)-\cdots}\frac{\pi}{2} \text{ if } p \text{ is even otherwise no } \frac{\pi}{2}$$

$$\therefore \text{ Given integral} = 4\left[a^3 b \cdot \frac{1\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2} + \frac{ab^3}{3}\cdot\frac{3\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2} \right]$$

$$= \frac{\pi}{4}(a^3 b + ab^3) = \frac{\pi}{4}ab(a^2+b^2).$$

Example 10. *Sketch the region of integration and evaluate the following integrals*

(*i*) $\int_1^{\log 8} \int_0^{\log y} e^{x+y}\, dxdy$

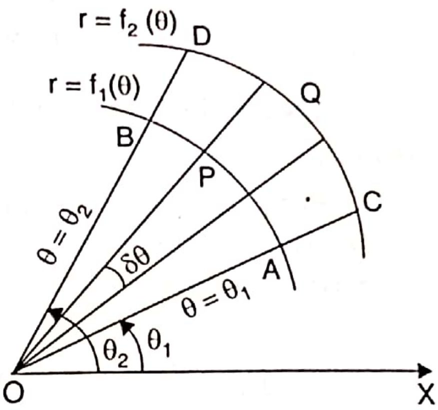
(P.T.U., May 2004)

(*ii*) *Evaluate* $\int_0^2 \int_0^{y^2} e^{x/y}\, dx\, dy$

6.3. EVALUATION OF DOUBLE INTEGRALS IN POLAR CO-ORDINATES

To evaluate $\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r,\theta)\,dr\,d\theta$ over the region bounded by the straight lines $\theta=\theta_1$, $\theta=\theta_2$ and the curves $r=r_1$, $r=r_2$, we first integrate w.r.t. r between the limits $r=r_1$ and $r=r_2$ (treating θ as a constant). The resulting expression is then integrated w.r.t. θ between the limits $\theta=\theta_1$ and $\theta=\theta_2$.

Geometrically, AB and CD are the curves $r=f_1(\theta)$ and $r=f_2(\theta)$ bounded by the lines $\theta=\theta_1$ and $\theta=\theta_2$ so that ACDB is the region of integration. PQ is a wedge of angular thickness $\delta\theta$. Then $\int_{r=r_1}^{r=r_2} f(r,\theta)\,dr$ indicates that the integration is performed along PQ (*i.e.*, r varies, θ is constant) and the integration w.r.t. θ means rotation of this strip PQ from AC to BD.



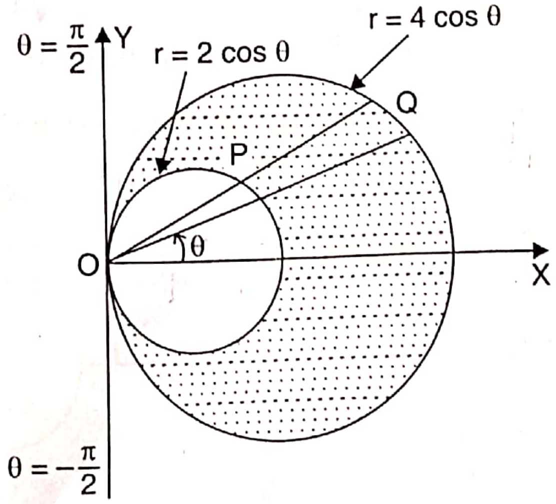
ILLUSTRATIVE EXAMPLES

Example 1. *Evaluate* $\int_0^{\pi/2}\left[\int_0^{a\cos\theta} r\sqrt{a^2-r^2}\,dr\right]d\theta.$

Sol.
$$I=\int_0^{\pi/2}\left[\int_0^{a\cos\theta} -\frac{1}{2}(a^2-r^2)^{1/2}(-2r)\,dr\right]d\theta.$$
$$=\int_0^{\pi/2}\left[-\frac{1}{2}\cdot\frac{(a^2-r^2)^{3/2}}{3/2}\right]_0^{a\cos\theta}d\theta$$
$$=-\frac{1}{3}\int_0^{\pi/2}(a^3\sin^3\theta-a^3)\,d\theta=-\frac{a^3}{3}\left[\frac{2}{3}-\frac{\pi}{2}\right]=\frac{a^3}{18}(3\pi-4).$$

Example 2. *Evaluate* $\iint r^3\,dr\,d\theta$, *over the area bounded between the circles* $r=2\cos\theta$ *and* $r=4\cos\theta$.

Sol. The region of integration R is shown shaded. Here r varies from $2\cos\theta$ to $4\cos\theta$ while θ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.



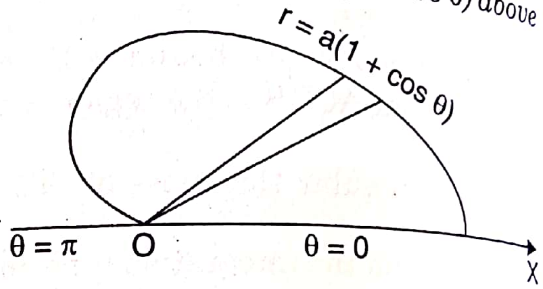
$$\therefore\ \iint_R r^3\,dr\,d\theta=\int_{-\pi/2}^{\pi/2}\int_{2\cos\theta}^{4\cos\theta} r^3\,dr\,d\theta$$
$$=\int_{-\pi/2}^{\pi/2}\left[\frac{r^4}{4}\right]_{2\cos\theta}^{4\cos\theta}d\theta$$
$$=\int_{-\pi/2}^{\pi/2}\frac{1}{4}(256\cos^4\theta-16\cos^4\theta)\,d\theta$$

$$= 60 \int_{-\pi/2}^{\pi/2} \cos^4 \theta \, d\theta$$

$$= 120 \int_0^{\pi/2} \cos^4 \theta \, d\theta$$

$$= 120 \times \frac{3 \times 1}{4 \times 2} \cdot \frac{\pi}{2} = \frac{45}{2} \pi.$$

Example 3. *Evaluate* $\iint r \sin \theta \, dr \, d\theta$ *over the area of the cardioid* $r = a(1 + \cos \theta)$ *above the initial line.*



Sol. The region of integration R is covered by radial strips whose ends are $r = 0$ and $r = a(1 + \cos \theta)$, the strips starting from $\theta = 0$ and ending at $\theta = \pi$.

$$\therefore \quad \iint_R r \sin \theta \, dr \, d\theta = \int_0^{\pi} \int_0^{a(1+\cos\theta)} r \sin \theta \, dr \, d\theta.$$

$$= \int_0^{\pi} \sin \theta \left[\frac{r^2}{2} \right]_0^{a(1+\cos\theta)} d\theta = \frac{1}{2} \int_0^{\pi} \sin \theta \cdot a^2 (1 + \cos \theta)^2 \, d\theta$$

$$= \frac{a^2}{2} \int_0^{\pi} 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cdot \left(2 \cos^2 \frac{\theta}{2} \right)^2 d\theta = 4a^2 \int_0^{\pi} \sin \frac{\theta}{2} \cos^5 \frac{\theta}{2} \, d\theta$$

$$= 4a^2 \int_0^{\pi/2} 2 \sin \phi \cos^5 \phi \, d\phi \qquad \left[\text{Putting } \frac{\theta}{2} = \phi \text{ and } d\theta = 2d\phi \right]$$

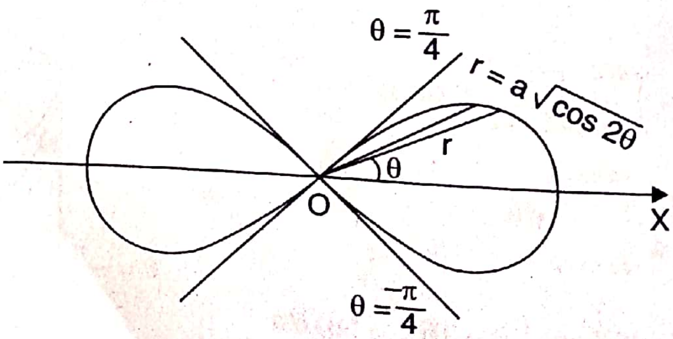
$$= -8a^2 \int_0^{\pi/2} \cos^5 \phi (-\sin \phi) \, d\phi = -8a^2 \cdot \left[\frac{\cos^6 \phi}{6} \right]_0^{\pi/2}$$

$$\because \int [f(x)]^n f'(x) \, dx = \frac{[f(x)]^{n+1}}{n+1}, n \neq -1$$

$$= -\frac{4a^2}{3}(0 - 1) = \frac{4a^2}{3}.$$

Example 4. *Evaluate* $\iint \frac{r \, dr \, d\theta}{\sqrt{a^2 + r^2}}$ *over one loop of the lemniscate* $r^2 = a^2 \cos 2\theta$.

Sol. The region of integration R is covered by radial strips whose ends are $r = 0$ and $r = a\sqrt{\cos 2\theta}$, the strips starting from $\theta = -\frac{\pi}{4}$ and ending at $\theta = \frac{\pi}{4}$.



$$\therefore \iint_R \frac{r\,dr\,d\theta}{\sqrt{a^2+r^2}} = \int_{-\pi/4}^{\pi/4}\int_0^{a\sqrt{\cos 2\theta}} \tfrac{1}{2}(a^2+r^2)^{-1/2}\cdot 2r\,dr\,d\theta$$

$$= \int_{-\pi/4}^{\pi/4}\left[\frac{1}{2}\cdot\frac{(a^2+r^2)^{1/2}}{1/2}\right]_0^{a\sqrt{\cos 2\theta}} d\theta = \int_{-\pi/4}^{\pi/4}[(a^2+a^2\cos 2\theta)^{1/2}-a]\,d\theta$$

$$= a\int_{-\pi/4}^{\pi/4}[(1+\cos 2\theta)^{1/2}-1]\,d\theta = a\int_{-\pi/4}^{\pi/4}[(2\cos^2\theta)^{1/2}-1]\,d\theta$$

$$= a\int_{-\pi/4}^{\pi/4}(\sqrt{2}\cos\theta-1)\,d\theta = 2a\int_0^{\pi/4}(\sqrt{2}\cos\theta-1)\,d\theta$$

| $\because$ integrand is an even function

$$= 2a\left[\sqrt{2}\sin\theta-\theta\right]_0^{\pi/4} = 2a\left[\sqrt{2}\cdot\frac{1}{\sqrt{2}}-\frac{\pi}{4}\right] = 2a\left(1-\frac{\pi}{4}\right).$$

TEST YOUR KNOWLEDGE

Evaluate the following integrals (1–4):

1. $\int_0^{\pi}\int_0^{a\sin\theta} r\,dr\,d\theta$
2. $\int_0^{\pi/2}\int_0^{a\cos\theta} r\sin\theta\,dr\,d\theta$
3. $\int_0^{\pi/2}\int_{a(1+\cos\theta)}^{a} r\,dr\,d\theta$
4. $\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^2\cos\theta\,dr\,d\theta$
5. Show that $\iint_R r^2\sin\theta\,dr\,d\theta = \frac{2a^3}{3}$, where R is the region bounded by the semi-circle $r = 2a\cos\theta$, above the initial line.
6. Evaluate $\iint r^3\,dr\,d\theta$ over the area included between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$.

Answers

1. $\frac{\pi a^2}{4}$
2. $\frac{a^2}{6}$
3. $-a^2\left(1+\frac{\pi}{8}\right)$
4. $\frac{5}{8}\pi a^3$
6. $\frac{45\pi}{2}$.

6.4. CHANGE OF ORDER OF INTEGRATION

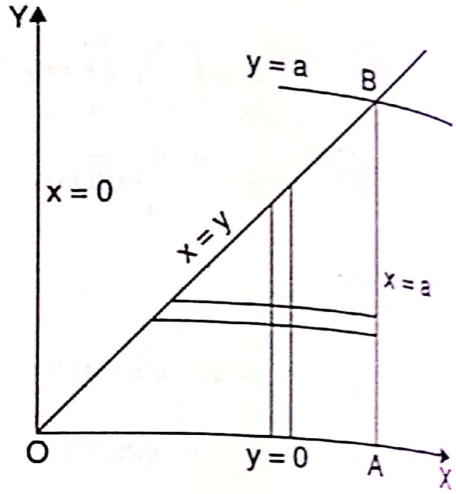
In a double integral, if the limits of integration are constant, then the order of integration is immaterial, provided the limits of integration are changed accordingly. Thus

$$\int_c^d\int_a^b f(x,y)\,dx\,dy = \int_a^b\int_c^d f(x,y)\,dy\,dx.$$

But if the limits of integration are variable, a change in the order of integration necessitates change in the limits of integration. A rough sketch of the region of integration helps in fixing the new limits of integration.

ILLUSTRATIVE EXAMPLES

Example 1. *Change the order of integration in* $\int_0^a \int_y^a \frac{x\,dx\,dy}{x^2+y^2}$ *and hence evaluate the same.*



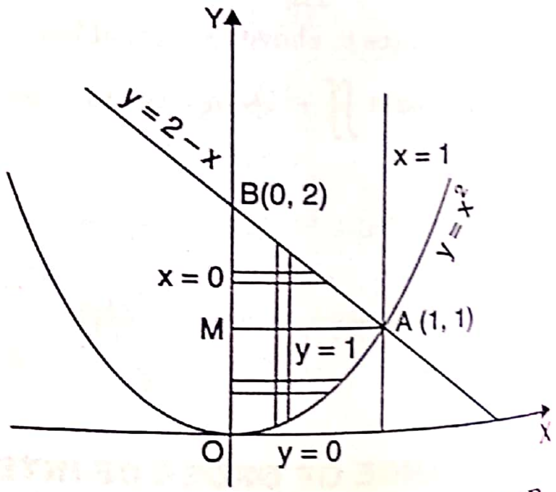
Sol. From the limits of integration, it is clear that the region of integration is bounded by $x = y$, $x = a$, $y = 0$ and $y = a$. Thus the region of integration is the ΔOAB and is divided into horizontal strips. For changing the order of integration, we divide the region of integration into vertical strips. The new limits of integration become: y varies from 0 to x and x varies from 0 to a.

$$\int_0^a \int_y^a \frac{x\,dx\,dy}{x^2+y^2} = \int_0^a \int_0^x \frac{x\,dy\,dx}{x^2+y^2}$$

$$= \int_0^a x \cdot \left[\frac{1}{x}\tan^{-1}\frac{y}{x}\right]_0^x dx$$

$$= \int_0^a \frac{\pi}{4}dx = \frac{\pi}{4}\cdot\left[x\right]_0^a = \frac{\pi a}{4}.$$

Example 2. *Change the order of integration in* $I = \int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx$ *and hence evaluate the same.* **(P.T.U., May 2006, May 2009, Dec. 2011, 2015)**



Sol. From the limits of integration, it is clear that we have to integrate first with respect to y which varies from $y = x^2$ to $y = 2 - x$ and then with respect to x which varies from $x = 0$ to $x = 1$. The region of integration OAB is divided into vertical strips. For changing the order of integration, we divide the region of integration into horizontal strips.

Solving $y = x^2$ and $y = 2 - x$, the co-ordinates of A are (1, 1). Draw AM $\perp$ OY. The region of integration is divided into two parts, OAM and MAB.

For the region OAM, x varies from 0 to $\sqrt{y}$ and y varies from 0 to 1. For the region MAB, x varies from 0 to $2 - y$ and y varies from 1 to 2.

$$\therefore \quad \int_0^1 \int_{x^2}^{2-x} xy\,dx\,dy = \int_0^1 \int_0^{\sqrt{y}} xy\,dx\,dy + \int_1^2 \int_0^{2-y} xy\,dx\,dy$$

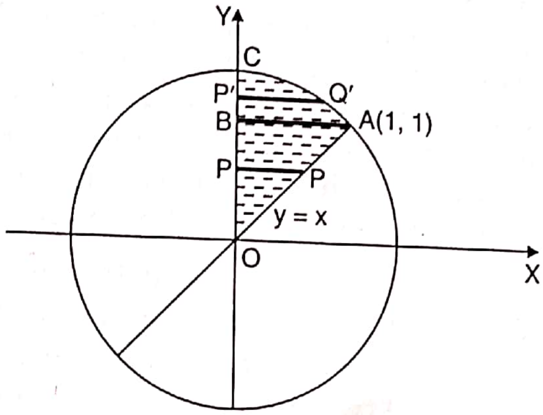
$$= \int_0^1 y\cdot\left[\frac{x^2}{2}\right]_0^{\sqrt{y}} dy + \int_1^2 y\cdot\left[\frac{x^2}{2}\right]_0^{2-y} dy = \frac{1}{2}\int_0^1 y^2 dy + \frac{1}{2}\int_1^2 y(2-y)^2\, dy$$

$$= \frac{1}{2}\cdot\left[\frac{y^3}{3}\right]_0^1 + \frac{1}{2}\int_1^2 (4y - 4y^2 + y^3)\, dy = \frac{1}{6} + \frac{1}{2}\left[2y^2 - \frac{4y^3}{3} + \frac{y^4}{4}\right]_1^2$$

$$= \frac{1}{6} + \frac{1}{2}\left[\left(8 - \frac{32}{3} + 4\right) - \left(2 - \frac{4}{3} + \frac{1}{4}\right)\right] = \frac{3}{8}.$$

Example 3. *Evaluate by changing the order of integration of* $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x\, dx\, dy}{\sqrt{x^2+y^2}}$.

Sol. From limits of integration it is clear that we have to first integrate w.r.t. y and then w.r.t. x. y varies from $y = x$ to $y = \sqrt{2-x^2}$ *i.e.*, $y^2 = 2 - x^2$ or $x^2 + y^2 = 2$ *i.e.*, a circle with centre at (0, 0) and radius $= \sqrt{2}$ and x varies 0 to 1. This integration is firstly performed along vertical strips and then along horizontal strips. For change of the order of integration we have to first perform integration along horizontal strips and then along vertical strips. The region of integration is shown shaded in the figure.



For horizontal strips the whole region is divided into two portion OAB and ACB. Let the horizontal strip in the portion OAB be PQ and that of in ACB be P′Q′.

The curve and the line intersect at A(1, 1).

For OAB ; x varies from 0 to x and y varies for 0 to 1.

For ACB ; x varies from 0 to $\sqrt{2-y^2}$ and y varies for 1 to $\sqrt{2}$

$$\therefore \int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x\, dx\, dy}{\sqrt{x^2+y^2}} = \int_0^1 \int_0^y \frac{x}{\sqrt{x^2+y^2}}\, dx\, dy + \int_0^{\sqrt{2}} \int_0^{\sqrt{2-y^2}} \frac{x\, dx\, dy}{\sqrt{x^2+y^2}}$$

$$= \frac{1}{2}\int_0^1 \int_0^y (x^2+y^2)^{-1/2}\,(2x)\, dx\, dy + \frac{1}{2}\int_1^{\sqrt{2}} \int_0^{\sqrt{2-y^2}} (2x)\,(x^2+y^2)^{-1/2}\, dx\, dy$$

$$= \frac{1}{2}\int_0^1 \left.\frac{(x^2+y^2)^{1/2}}{1/2}\right|_0^y dy + \frac{1}{2}\int_1^{\sqrt{2}} \left.\frac{(x^2+y^2)^{1/2}}{1/2}\right|_0^{\sqrt{2-y^2}} dy$$

| By using $\int [f(x)]\, f'(x) dx = \frac{[f(x)]^{n+1}}{n+1},\ n \neq -1$

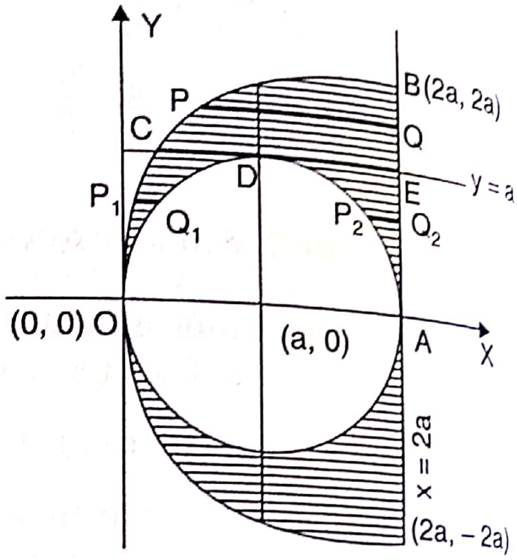
$$= \int_0^1 \left(\sqrt{2}y - y\right) dy + \int_1^{\sqrt{2}} \left(\sqrt{2} - y\right) dy$$

$$= \left(\frac{\sqrt{2}y^2}{2} - \frac{y^2}{2}\right)\Bigg|_0^1 + \left(\sqrt{2}y - \frac{y^2}{2}\right)\Bigg|_1^{\sqrt{2}} = \frac{\sqrt{2}-1}{2} + 2 - 1 - \sqrt{2} + \frac{1}{2}$$

$$= \frac{1}{2}\left[\sqrt{2} - 1 + 4 - 2 - 2\sqrt{2} + 1\right] = \frac{1}{2}\left[2 - \sqrt{2}\right] = 1 - \frac{1}{\sqrt{2}}.$$

Example 4. *Change the order of integration of*

$$\int_0^{2a}\int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} V\,dy\,dx.$$ **(P.T.U., Dec. 2003)**



Sol. From the limits of integration it is clear that we have to first integrate w.r.t. y which varies from $\sqrt{2ax-x^2}$ to $\sqrt{2ax}$ *i.e.*, $\sqrt{2ax-x^2} \le y \le \sqrt{2ax}$ *i.e.*, $2ax - x^2 \le y^2 \le 2ax$ and then w.r.t. x which varies from 0 to $2a$ *i.e.*, $0 \le x \le 2a$. This shows that integration is firstly performed along veritcal strips. For change of the order of integration we have to perform integration firstly along horizontal strips and then along vertical. We draw rough sketch of the region $y^2 = 2ax - x^2$ or $x^2 + y^2 - 2ax = 0$ which is a circle with centre at $(a, 0)$ and radius a, $y = \sqrt{2ax}$ or $y^2 = 2ax$ is a right handed parabola. The region of integration is shown shaded in the figure.

$x = 0$ represents y-axis and $x = 2a$ represents a line parallel to y-axis.

The two curves intersect at (0, 0). For horizontal strip PQ we see that only the region BCE is covered ∴ we divide the whole region into three portions namely: BCE, ODC and AED

for BCE;

x varies from $\dfrac{y^2}{2a}$ to $2a$

y varies from a to $2a$

for ODC ; x varies from $\dfrac{y^2}{2a}$ to $a + \sqrt{a^2 - y^2}$

y varies from 0 to a

for AED x varies from $a + \sqrt{a^2 - y^2}$ to $2a$

y varies from 0 to a

$$\because\ y^2 = 2ax - x^2 \text{ or } x^2 - 2ax + y^2 = 0$$

$$x = \frac{2a \pm \sqrt{4a^2 - 4y^2}}{2} = a \pm \sqrt{a^2 - y^2}$$

$$\therefore \int_0^{2a}\int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} V\,dy\,dx = \int_a^{2a}\int_{y^2/2a}^{2a} V\,dx\,dy + \int_0^a\int_{y^2/2a}^{a+\sqrt{a^2-y^2}} V\,dx\,dy + \int_0^a\int_{a+\sqrt{a^2-y^2}}^{2a} V\,dx\,dy$$

Example 5. *Change the order of integration in the following integral and evaluate:*

(*i*) $\displaystyle\int_0^a\int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} dx\,dy$ (*ii*) $\displaystyle\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\,dx.$ **(P.T.U., Jan. 2010)**

Sol. (*i*) $\displaystyle\int_0^a\int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} dx\,dy$

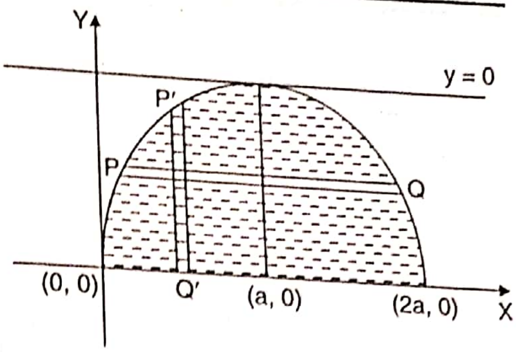
From the limits of integration it is clear that we have to first integrate w.r.t. x and then w.r.t. y

x varies from $a - \sqrt{a^2 - y^2}$ to $a + \sqrt{a^2 - y^2}$

i.e., $x = a \pm \sqrt{a^2 - y^2}$; $(x-a)^2 = a^2 - y^2$

or $x^2 + y^2 - 2ax = 0$

i.e., inside the circle with centre at $(a, 0)$ and radius a and y varies from 0 to a.



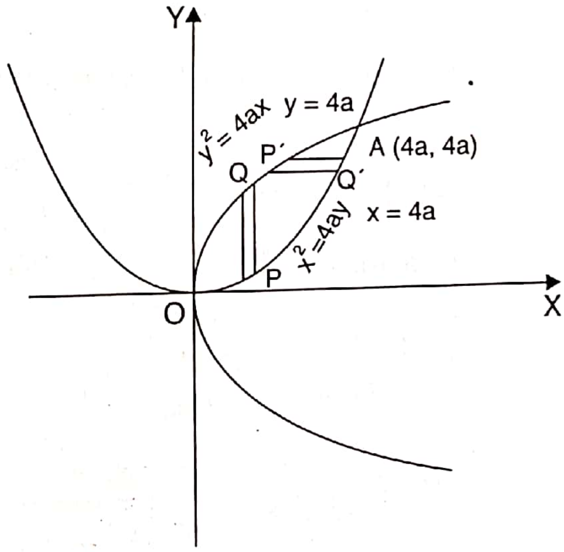
This integration is first perfomed along horizontal strips PQ. For changing the order of integration divide the region into vertical strips P′Q′ where y varies from 0 to $\sqrt{2ax - x^2}$ and x varies from 0 to $2a$. The region of integration is shown shaded in the figure.

$$\therefore \quad \int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} dxdy = \int_0^{2a} \int_0^{\sqrt{2ax-x^2}} dy\, dx = \int_0^{2a} y \Big|_0^{\sqrt{2ax-x^2}} dx$$

$$= \int_0^{2a} \sqrt{2ax - x^2}\, dx = \int_0^{2a} \sqrt{a^2 - (x-a)^2}\, dx$$

$$= \frac{(x-a)\sqrt{a^2-(x-a)^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x-a}{a} \Big|_0^{2a} = \frac{a^2}{2}\frac{\pi}{2} - \frac{a^2}{2}\left(-\frac{\pi}{2}\right) = \frac{a^2\pi}{2}.$$

(*ii*) From the limits of integration, it is clear that we have to integrate first w.r.t. y which varies from $y = \frac{x^2}{4a}$ to $y = 2\sqrt{ax}$ and then w.r.t. x which varies from $x = 0$ to $x = 4a$. Thus integration is first performed along the vertical strip PQ which extends from a point P on the parabola $y = \frac{x^2}{4a}$ (*i.e.*, $x^2 = 4ay$) to the point Q on the parabola $y = 2\sqrt{ax}$ (*i.e.*, $y^2 = 4ax$). Then the strip slides from O to A $(4a, 4a)$, the point of intersection of the two parabolas.



For changing the order of integration, we divide the region of integration OPAQO into horizontal strips P′Q′ which extend from P′ on the parabola $y^2 = 4ax$ *i.e.*, $x = \frac{y^2}{4a}$ to Q′ on the parabola $x^2 = 4ay$ *i.e.*, $x = 2\sqrt{ay}$. Then this strip slides from O to A$(4a, 4a)$, *i.e.*, y varies from 0 to $4a$.

$$\therefore \quad \int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dy\, dx = \int_0^{4a} \int_{y^2/4a}^{2\sqrt{ay}} dx\, dy$$

$$= \int_0^{4a} \Big[x \Big]_{y^2/4a}^{2\sqrt{ay}} dy = \int_0^{4a} \left(2\sqrt{ay} - \frac{y^2}{4a} \right) dy$$

$$= \left[2\sqrt{a} \cdot \frac{y^{3/2}}{3/2} - \frac{y^3}{12a} \right]_0^{4a} = \frac{4}{3}\sqrt{a} \cdot (4a)^{3/2} - \frac{64a^3}{12a}$$

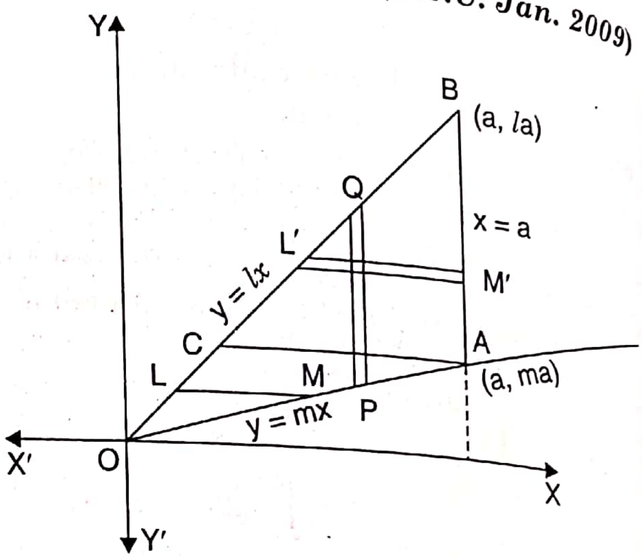
$$= \frac{4}{3}\sqrt{a} \cdot 8a^{3/2} - \frac{16a^2}{3} = \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16a^2}{3}.$$

Example 6. *Change the order of integration in* $\int_0^a \int_{mx}^{lx} f(x, y)\, dy\, dx$.

(P.T.U. Jan. 2009)

Sol. From the limits of integration, it is clear that we have to integrate first w.r.t. y which varies from $y = mx$ to $y = lx$ and then w.r.t. x which varies from $x = 0$ to $x = a$.

Thus integration is first performed along the vertical strip PQ which extends from a point P on the line $y = mx$ to the point Q on the line $y = lx$.



For change of order of integration we have to first perform integration along horizontal strips and then along vertical strips.

For horizontal strips the whole region is divided into two portions OAC and CAB.

Let the horizontal strip in the portion OAC be LM and that of in CAB be L′M′. For the region OAC, x varies from $\frac{y}{l}$ to $\frac{y}{m}$ and y varies from 0 to ma.

For the region CAB; x varies from $\frac{y}{l}$ to a and y varies from ma to la

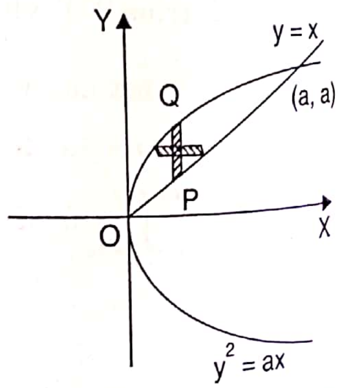
$$\therefore \quad \int_0^a \int_{mx}^{lx} f(x, y)\, dy\, dx = \int_{y=0}^{ma} \int_{x=\frac{y}{l}}^{y/m} f(x, y)\, dx\, dy + \int_{y=ma}^{la} \int_{x=\frac{y}{l}}^{a} f(x, y)\, dx\, dy.$$

Example 7. *Evaluate the following*

(*i*) $\displaystyle\int_0^a \int_{y^2/a}^{y} \frac{y\, dx\, dy}{(a-x)\sqrt{ax-y^2}}$ (*ii*) $\displaystyle\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx$. **(P.T.U., May 2004, Dec., 2011)**

Sol. (*i*) $\displaystyle\int_0^a \int_{y^2/a}^{y} \frac{y\, dx\, dy}{(a-x)\sqrt{ax-y^2}}$

Here limits of integration are given by $0 \le y \le a$ and $\frac{y^2}{a} \le x \le y$. The limits of x are depending on y where as that of y are constant. $\therefore$ We have to integrate first w.r.t. x and then w.r.t. y but we cannot easily integrate in this order, integral of $\frac{y}{(a-x)\sqrt{ax-y^2}}$ w.r.t. y is simpler.



$\therefore$ We change the order of integration for this we first sketch the region given by $x = \frac{y^2}{a}$ or $y^2 = ax$ (a right handed parabola) and $x = y$ is a straight line and these two intersect at points given by $x^2 = ax$ or $x = 0$, $x = a$ *i.e.*, at (0, 0) and (a, a).

To change the order of integration we need limits of y as variable and that of x as constant.

$\therefore$ We divide the region into vertical strips PQ

$\therefore$ y varies from x to $\sqrt{ax}$

and x varies from 0 to a

$$\therefore \int_0^a \int_{y^2/a}^{y} \frac{y\,dx\,dy}{(a-x)\sqrt{ax-y^2}} = \int_0^a \left[\int_x^{\sqrt{ax}} \frac{y}{(a-x)\sqrt{ax-y^2}}\,dy\right] dx$$

$$= \int_0^a \left[\frac{1}{a-x}\int_x^{\sqrt{ax}} \left(-\frac{1}{2}\right)(-2y)(ax-y^2)^{-1/2}\,dy\right] dx$$

$$= \int_0^a \frac{1}{a-x}\left(-\frac{1}{2}\right)\left[\frac{(ax-y^2)^{1/2}}{1/2}\Bigg|_x^{\sqrt{ax}}\right] dx = \int_0^a \frac{-1}{2(a-x)}[(0-2(ax-x^2)^{1/2}]\,dx$$

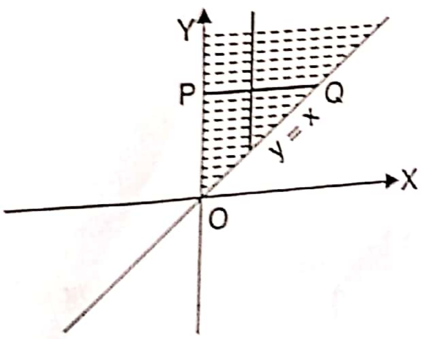
$$= \int_0^a \frac{1}{a-x}\sqrt{x}\,\sqrt{a-x}\,dx = \int_0^a \frac{\sqrt{x}}{\sqrt{a-x}}\,dx$$

Put $x = a\sin^2\theta$; $dx = 2a\sin\theta\cos\theta\,d\theta$

$$= \int_0^{\pi/2} \frac{\sqrt{a}\sin\theta}{\sqrt{a}\cos\theta}\,2a\sin\theta\cos\theta\,d\theta = 2a\int_0^{\pi/2}\sin^2\theta\,d\theta$$

$$= 2a\,\frac{1}{2}\,\frac{\pi}{2} = \frac{a\pi}{2}.$$

(ii) $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$



Here the limits of integrations of y are from x to ∞ and the limits of integrations of x are from 0 to ∞. As $\frac{e^{-y}}{y}$ cannot be integrated w.r.t. y, $\therefore$ we change the order of integration. So we first sketch the region and then divide it into horizontal strips (Region is the shaded portion).

For horizontal strip x varies from 0 to y and y varies from 0 to ∞

$$\therefore \int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dydx = \int_0^\infty \int_0^y \frac{e^{-y}}{y}\,dx\,dy = \int_0^\infty \left[\int_0^y \frac{e^{-y}}{y}\,dx\right] dy$$

$$= \int_0^\infty \frac{e^{-y}}{y}\,x\Big|_0^y\,dy = \int_0^\infty \frac{e^{-y}}{y}\cdot y\,dy = \int_0^\infty e^{-y}\,dy = -e^{-y}\Big|_0^\infty = 0+1 = 1.$$

6.5. TRIPLE INTEGRALS

Consider a function $f(x, y, z)$ which is continuous at every point of a finite region V of three dimensional space. Divide the region V into n sub-regions of respective volumes $\delta V_1, \delta V_2, \ldots\ldots, \delta V_n$. Let (x_r, y_r, z_r) be an arbitrary point in the rth sub-region. Consider the sum $\sum_{r=1}^{n} f(x_r, y_r, z_r)\,\delta V_r$.

The limit of this sum as $n \to \infty$ and $\delta V_r \to 0$, if it exists, is called the *triple integral* of $f(x, y, z)$ over the region V and is denoted by $\iiint_V f(x, y, z)\,dV$.

For purposes of evaluation, it can be expressed as the repeated integral

$$\int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y,z)\,dx\,dy\,dz, \qquad \ldots(1)$$

the order of integration depending upon the limits.

Let x_1, x_2 be function of y, z ; y_1, y_2 be function of z and z_1, z_2 be constants, *i.e.*,

Let $x_1 = f_1(y, z)$, $x_2 = f_2(y, z)$, $y_1 = \phi_1(z)$, $y_2 = \phi_2(z)$ and $z_1 = a$, $z_2 = b$.

Then the integral (1) is evaluated as follows:

$$\int_{z_1=a}^{z_2=b}\left[\int_{y_1=\phi_1(z)}^{y_2=\phi_2(z)}\left\{\int_{x_1=f_1(y,z)}^{x_2=f_2(y,z)} f(x,y,z)\,dx\right\}dy\right]dz$$

First $f(x, y, z)$ is integrated w.r.t. x (keeping y and z constant) between the limits x_1 and x_2. The resulting expression, which is a function of y and z is then integrated w.r.t. y (keeping z constant) between the limits y_1 and y_2. The resulting expression, which is a function of z only is then integrated w.r.t. z between the limits z_1 and z_2. The order of integration is from the innermost rectangle to the outermost rectangle.

Limits involving two variables are kept innermost, then the limits involving one variable and finally the constant limits.

If x_1, x_2 ; y_1, y_2 and z_1, z_2 are all constants, then the order of integration is immaterial, provided the limits are changed accordingly. Thus

$$\int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y,z)\,dx\,dy\,dz = \int_{x_1}^{x_2}\int_{z_1}^{z_2}\int_{y_1}^{y_2} f(x,y,z)\,dy\,dz\,dx.$$

$$= \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x,y,z)\,dz\,dy\,dx, \text{ etc.}$$

ILLUSTRATIVE EXAMPLES

Example 1. *Evaluate* $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}}\, dz\, dy\, dx.$ **(P.T.U., May 2004)**

Sol.
$$I = \int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{(1-x^2-y^2)-z^2}}\, dz\, dy\, dx$$
$$= \int_0^1\int_0^{\sqrt{1-x^2}}\left[\sin^{-1}\frac{z}{\sqrt{1-x^2-y^2}}\right]_0^{\sqrt{1-x^2-y^2}} dy\, dx$$
$$= \int_0^1\int_0^{\sqrt{1-x^2}} \frac{\pi}{2}\, dy\, dx = \int_0^1 \frac{\pi}{2}\Big[y\Big]_0^{\sqrt{1-x^2}} dx$$
$$= \frac{\pi}{2}\int_0^1 \sqrt{1-x^2}\, dx = \frac{\pi}{2}\left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1} x\right]_0^1 = \frac{\pi}{4}[\sin^{-1} 1] = \frac{\pi}{4}\cdot\frac{\pi}{2} = \frac{\pi^2}{8}.$$

Example 2. *Evaluate* $\int_0^2\int_1^2\int_0^{yz} xyz\, dx\, dy\, dz.$ **(P.T.U., May 2006, May 2009)**

Sol. $\int_0^2\int_1^2\left[\int_0^{yz} xyz\, dx\right] dy\, dz$
$$= \int_0^2\int_1^2 yz\, \frac{x^2}{2}\Bigg|_0^{yz} dy\, dz = \int_0^2\left[\int_1^2 \frac{y^3z^3}{2}\, dy\right] dz$$
$$= \int_0^2 \frac{z^3}{2}\cdot\frac{y^4}{4}\Bigg|_1^2 dz = \int_0^2 \frac{z^3}{8}(16-1)\, dz = \frac{15}{8}\cdot\frac{z^4}{4}\Bigg|_0^2$$
$$= \frac{15}{8}\cdot\frac{16}{4} = \frac{15}{2}.$$

Example 3. *Evaluate* $\int_1^e\int_1^{\log y}\int_1^{e^x} \log z\, dz\, dx\, dy.$

Sol.
$$I = \int_1^e\int_1^{\log y}\left[\int_1^{e^x} \log z\, dz\right] dx\, dy$$

Since $\int_1^{e^x} \log z\, dz = \int_1^{e^x} \log z \cdot 1\, dz$

Integrating by parts $= \Big[\log z \cdot z\Big]_1^{e^x} - \int_1^{e^x} \frac{1}{z}\cdot z\, dz$

$$= e^x \log e^x - 0 - \Big[z\Big]_1^{e^x} = xe^x - e^x + 1 = (x-1)\, e^x + 1$$

$$\therefore \quad I = \int_1^e \int_1^{\log y} [(x-1)\, e^x + 1]\, dx\, dy$$

Now $\int_1^{\log y} [(x-1)\, e^x + 1]\, dx = \int_1^{\log y} (x-1)\, e^x\, dx + \Big[x\Big]_1^{\log y}$

$$= \Big[(x-1)\, e^x\Big]_1^{\log y} - \int_1^{\log y} 1 \cdot e^x\, dx + \log y - 1 \qquad \text{(Integration by parts)}$$

$$= (\log y - 1)\, e^{\log y} - \Big[e^x\Big]_1^{\log y} + \log y - 1$$

$$= y(\log y - 1) - (e^{\log y} - e) + \log y - 1 \qquad [\because \; e^{\log y} = y]$$

$$= y(\log y - 1) - y + e + \log y - 1 = (y+1) \log y - 2y + e - 1$$

$$\therefore \quad I = \int_1^e [\log y \cdot (y+1) - 2y + e - 1]\, dy$$

$$= \left[\log y \cdot \left(\frac{y^2}{2} + y\right)\right]_1^e - \int_1^e \frac{1}{y}\left(\frac{y^2}{2} + y\right) dy - \Big[y^2\Big]_1^e + (e-1)\Big[y\Big]_1^e$$

$$= \frac{e^2}{2} + e - \int_1^e \left(\frac{y}{2} + 1\right) dy - (e^2 - 1) + (e-1)^2$$

$$= \frac{e^2}{2} + e - \left[\frac{y^2}{4} + y\right]_1^e - 2e + 2 = \frac{e^2}{2} + e - \left[\left(\frac{e^2}{4} + e\right) - \left(\frac{1}{4} + 1\right)\right] - 2e + 2$$

$$= \frac{e^2}{4} - 2e + \frac{13}{4} = \frac{1}{4}(e^2 - 8e + 13).$$

Example 4. *Evaluate* $\int_0^{\log 2} \int_0^x \int_0^{x+\log y} e^{x+y+z}\, dz\, dy\, dx.$

Sol. $\int_0^{\log 2} \int_0^x \int_0^{x+\log y} e^{x+y} \cdot e^z\, dz\, dy\, dx = \int_0^{\log 2} \int_0^x e^{x+y} \cdot e^z \Big|_0^{x+\log y} dy\, dx$

$$= \int_0^{\log 2} \int_0^x e^{x+y} (e^{x+\log y} - 1)\, dy\, dx$$

$$= \int_0^{\log 2} \int_0^x e^{x+y} (e^x \cdot e^{\log y} - 1) dy\, dx = \int_0^{\log 2} \int_0^x e^{x+y} (ye^x - 1)\, dy\, dx$$

$$= \int_0^{\log 2} \int_0^x [e^{2x}(ye^y) - e^x \cdot e^y]\, dy\, dx$$

$$= \int_0^{\log 2} \left\{ e^{2x} (y-1)\, e^y - e^x\, e^y \Big|_0^x \right\} dx$$

$$\Big| \because \int y\, e^y\, dy = y\, e^{2y} - \int 1 \cdot e^y\, dy = (y-1)\, e^y$$

$$= \int_0^{\log 2} \{e^{2x} (x-1)\, e^x - e^{2x} + e^{2x} + e^x\}\, dx$$

$$= \int_0^{\log 2} \{(x-1)\, e^{3x} + e^x\}\, dx$$

$$= (x-1)\frac{e^{3x}}{3}\Big|_0^{\log 2} - \int_0^{\log 2} 1\cdot\frac{e^{3x}}{3}\,dx + e^x\Big|_0^{\log 2}$$

$$= (\log 2 - 1)\cdot\frac{1}{3}e^{3\log 2} + \frac{1}{3} - \frac{e^{3x}}{9}\Big|_0^{\log 2} + (e^{\log 2} - 1)$$

$$= \frac{8}{3}(\log 2 - 1) + \frac{1}{3} - \frac{e^{3\log 2}}{9} + \frac{1}{9} + 2 - 1 \qquad \Big|\ \because\ e^{3\log 2} = e^{\log 2^3} = 8$$

$$= \frac{8}{3}\log 2 - \frac{8}{3} + \frac{1}{3} - \frac{8}{9} + \frac{1}{9} + 1 = \frac{8}{3}\log 2 - \frac{19}{9}.$$

Example 5. *Evaluate* $\int_0^{\frac{\pi}{2}}\int_0^{a\cos\theta}\int_0^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta.$

Sol. $\int_0^{\frac{\pi}{2}}\int_0^{a\cos\theta} rz\Big|_0^{\sqrt{a^2-r^2}} dr\,d\theta$

$$= \int_0^{\frac{\pi}{2}}\left[\int_0^{a\cos\theta} r\sqrt{a^2-r^2}\,dr\right]d\theta$$

$$= \int_0^{\frac{\pi}{2}}\left(-\frac{1}{2}\right)\frac{(a^2-r^2)^{3/2}}{3/2}\Big|_0^{a\cos\theta} d\theta \qquad \text{using } \int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1},\ n \neq -1$$

$$= -\frac{1}{3}\int_0^{\frac{\pi}{2}}(a^3\sin^3\theta - a^3)\,d\theta$$

$$= -\frac{a^3}{3}\left\{\int_0^{\frac{\pi}{2}}\sin^3\theta\,d\theta - \int_0^{\frac{\pi}{2}}1\cdot d\theta\right\}$$

$$= -\frac{a^3}{3}\left\{\frac{2}{3.1} - \frac{\pi}{2}\right\} \qquad \text{using } \int_0^{\frac{\pi}{2}}\sin^n\theta\,d\theta = \frac{(n-1)(n-3)\ldots}{n\,(n-1)\ldots} \text{ when } n \text{ is odd}$$

$$= \frac{a^3}{3}\left\{\frac{\pi}{2} - \frac{2}{3}\right\}.$$

Example 6. *Evaluate* $\iiint \frac{dx\,dy\,dz}{(x+y+z+1)^3}$ *over the tetrahedron bounded by the coordinate planes and the plane* $x+y+z=1.$ **(P.T.U., Dec. 2003)**

Sol. $\iiint \frac{dx\,dy\,dz}{(x+y+z+1)^3}$

$$= \int_0^1\int_0^{1-x}\int_0^{1-x-y}(x+y+z+1)^{-3}\,dz\,dy\,dx = \int_0^1\int_0^{1-x}\frac{(x+y+z+1)^{-2}}{-2}\Big|_0^{1-x-y} dy\,dx$$

$$= -\frac{1}{2}\int_0^1\int_0^{1-x}(x+y+1+1-x-y)^{-2} - (x+y+1)^{-2}]\,dy\,dx$$

$$= -\frac{1}{2}\int_0^1\int_0^{1-x}\frac{1}{2^2} - (x+y+1)^{-2}]\,dy\,dx$$

$$= -\frac{1}{2}\int_0^1 \frac{1}{4}y - \frac{(x+y+1)^{-1}}{-1}\Bigg|_0^{1-x} dx$$

$$= -\frac{1}{2}\int_0^1 \left[\frac{1-x}{4} + 2^{-1} - (x+1)^{-1}\right] dx$$

$$= -\frac{1}{2}\int_0^1 \left[\frac{3}{4} - \frac{x}{4} - \frac{1}{x+1}\right] dx$$

$$= -\frac{1}{2}\left[\frac{3}{4}x - \frac{x^2}{8} - \log(x+1)\Bigg|_0^1\right]$$

$$= -\frac{1}{2}\cdot\left(\frac{3}{4} - \frac{1}{8} - \log 2\right) = -\frac{1}{2}\left(\frac{5}{8} - \log 2\right) = \frac{1}{2}\log 2 - \frac{5}{16}.$$

TEST YOUR KNOWLEDGE

Evaluate the following integrals (1–10):

1. $\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a} (x^2+y^2+z^2)\,dx\,dy\,dz$ **(P.T.U., May 2015)**

2. $\int_0^a\int_0^a\int_0^a (yz+zx+xy)\,dx\,dy\,dz$

3. $\int_{-1}^{1}\int_0^{z}\int_{x-z}^{x+z} (x+y+z)\,dy\,dx\,dz.$ **(P.T.U., May 2014)**

4. $\int_1^3\int_{1/x}^{1}\int_0^{\sqrt{xy}} xyz\,dz\,dy\,dx$

5. $\int_0^1\int_0^{1-x}\int_0^{1-x-y} xyz\,dz\,dy\,dx$ **(P.T.U., Dec. 2013)**

6. $\int_0^1\int_{y^2}^{1-x}\int_0^{1-x} x\,dz\,dx\,dy$

7. $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xyz\,dz\,dy\,dx$ **(P.T.U., May 2012)**

8. $\int_0^{\pi/2} d\theta \int_0^{a\sin\theta} dr \int_0^{(a^2-r^2)} r dz$

9. $\int_0^a\int_0^x\int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx$

10. $\int_0^{\pi/2}\int_0^{a\sin\theta}\int_0^{\frac{a^2-r^2}{a}} r\,dz\,dr\,d\theta$

11. Evaluate $\iiint (x+y+z)\,dx\,dy\,dz$ over the tetrahedron bounded by $x=0, y=0, z=0$ and $x+y+z=1$
[**Hint.** See S.E. 6]

Answers

1. $\frac{8}{3}abc(a^2+b^2+c^2)$ 2. $\frac{3}{4}a^5$ 3. 0 4. $\frac{13}{9} - \frac{1}{6}\log 3$

5. $\frac{1}{720}$ 6. $\frac{4}{35}$ 7. $\frac{1}{48}$ 8. $\frac{5\pi a^3}{64}$

9. $\frac{1}{8}e^{4a} - \frac{3}{4}e^{2a} + e^a - \frac{3}{8}$ 10. $\frac{5\pi a^3}{64}$ 11. $\frac{1}{8}$.

6.6. CHANGE OF VARIABLES

Quite often, the evaluation of a double or triple integral is greatly simplified by a suitable change of variables.

Let the variables x, y in the double integral $\iint_R f(x, y)\, dx\, dy$ be changed to u, v by means of the relations $x = \phi(u, v)$, $y = \psi(u, v)$, then the double integral is transformed into

$$\iint_{R'} f\{\phi(u, v), \psi(u, v)\}\ |\,J\,|\ du\, dv \quad \text{where } J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

is the Jacobian of transformation from (x, y) to (u, v) co-ordinates and R' is the region in the uv-plane which corresponds to the region R in the xy-plane.

(*i*) *To change cartesian co-ordinates (x, y) to polar co-ordinates (r, θ).*

Here we have $x = r\cos\theta$, $y = r\sin\theta$ so that $x^2 + y^2 = r^2$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r$$

$$\therefore \iint_R f(x, y)\, dx\, dy = \iint_{R'} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

i.e., replace x by $r\cos\theta$, y by $r\sin\theta$ and $dxdy$ by $rdrd\theta$.

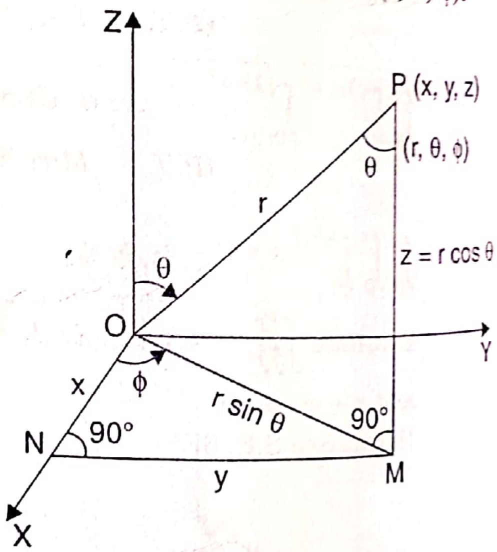
(*ii*) *To change cartesian co-ordinates (x, y, z) to spherical polar co-ordinates (r, θ, ϕ).*

Here, we have $x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

so that $x^2 + y^2 + z^2 = r^2$



$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta$$

$$\therefore \quad \iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_{V'} f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

i.e., to replace x by $r\sin\theta\cos\phi$; y by $r\sin\theta\sin\phi$; z by $r\cos\theta$ and $dx\,dy\,dz$ by $r^2\sin\theta\,dr\,d\theta\,d\phi$

(iii) To change cartesian co-ordinates (x, y, z) to cylindrical co-ordinates (r, φ, z).

Here we have $x = r\cos\phi$

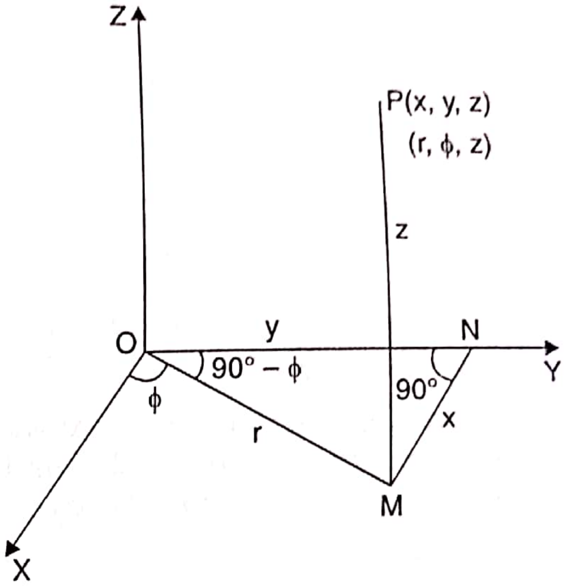
$y = r\sin\phi$

$z = z$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\phi & -r\sin\phi & 0 \\ \sin\phi & r\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= r(\cos^2\phi + \sin^2\phi) = r$$



$$\therefore \quad \iiint_V f(x, y, z)\,dx\,dy\,dz$$

$$= \iiint_{V'} f(r\cos\phi, r\sin\phi, z)\,r\,dr\,d\phi\,dz.$$

i.e., replace x by $r\cos\phi$, y by $r\sin\phi$, z remains same replace $dx\,dy$ by $r dr\,d\phi$; dz remain same.

ILLUSTRATIVE EXAMPLES

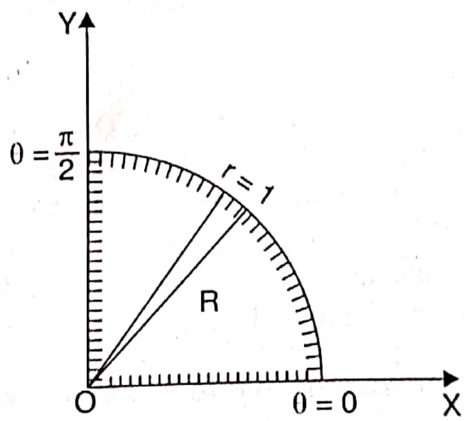
Example 1. *Evaluate* $\iint \sqrt{\dfrac{1-x^2-y^2}{1+x^2+y^2}}\,dx\,dy$ *over the positive quadrant of the circle* $x^2+y^2=1$.

Sol. Changing to polar coordinates by putting $x = r\cos\theta$, $y = r\sin\theta$; $x^2+y^2 = 1$ transforms into $r = 1$. For the region of integration R, r varies from 0 to 1 and θ varies from 0 to $\dfrac{\pi}{2}$.

$$I = \iint_R \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\,dx\,dy$$

$$= \int_0^{\pi/2}\int_0^1 \sqrt{\frac{1-r^2}{1+r^2}}\;r\,dr\,d\theta \quad |\,dx\,dy \text{ is replaced by } r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^1 \frac{r(1-r^2)}{\sqrt{1-r^4}}\,dr\,d\theta$$



Now $\displaystyle\int_0^1 \frac{r(1-r^2)}{\sqrt{1-r^4}}\,dr = \int_0^1 \left(\frac{r}{\sqrt{1-r^4}} - \frac{r^3}{\sqrt{1-r^4}}\right) dr$

$$= \frac{1}{2}\int_0^1 \frac{2r}{\sqrt{1-r^4}}\,dr + \frac{1}{4}\int_0^1 -4r^3(1-r^4)^{-1/2}dr$$

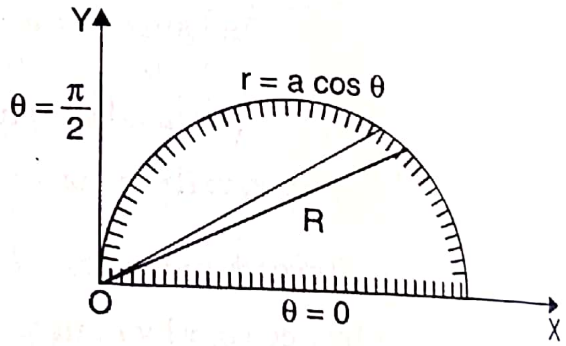
$$= \frac{1}{2}\int_0^1 \frac{dt}{\sqrt{1-t^2}} + \frac{1}{4}\cdot\left[\frac{(1-r^4)^{1/2}}{1/2}\right]_0^1, \text{ where } t = r^2$$

$$= \frac{1}{2}\left[\sin^{-1} t\right]_0^1 + \frac{1}{2}(0-1) = \frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{1}{2} = \frac{\pi}{4} - \frac{1}{2}$$

$$\therefore \quad I = \int_0^{\pi/2}\left(\frac{\pi}{4}-\frac{1}{2}\right)d\theta = \left(\frac{\pi}{4}-\frac{1}{2}\right)\left[\theta\right]_0^{\pi/2} = \left(\frac{\pi}{4}-\frac{1}{2}\right)\frac{\pi}{2} = \frac{\pi^2}{8} - \frac{\pi}{4}.$$

Example 2. *Evaluate* $\iint \sqrt{a^2-x^2-y^2}\; dx\,dy$ *over the semi-circle* $x^2 + y^2 = ax$ *in the positive quadrant.*

Sol. Changing to polar co-ordinates, $x^2 + y^2 = ax$ transforms into $r = a\cos\theta$. For the region of integration R, r varies from 0 to $a\cos\theta$ and θ varies from 0 to $\frac{\pi}{2}$.



$$\therefore \quad \iint_R \sqrt{a^2-x^2-y^2}\; dx\,dy$$

$$= \int_0^{\pi/2}\int_0^{a\cos\theta} \sqrt{a^2-r^2}\,.\,r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^{a\cos\theta} -\frac{1}{2}(a^2-r^2)^{1/2}(-2r)\,dr\,d\theta$$

$$= \int_0^{\pi/2} -\frac{1}{2}\cdot\left[\frac{(a^2-r^2)^{3/2}}{3/2}\right]_0^{a\cos\theta} d\theta$$

$$= -\frac{1}{3}\int_0^{\pi/2}(a^3\sin^3\theta - a^3)\,d\theta = -\frac{a^3}{3}\int_0^{\pi/2}(\sin^3\theta - 1)\,d\theta$$

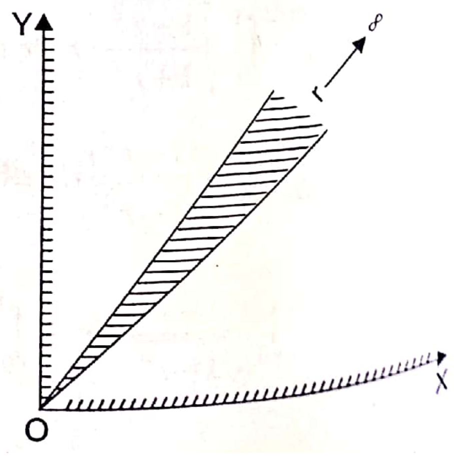
$$= -\frac{a^3}{3}\left[\frac{2}{3}-\frac{\pi}{2}\right] = \frac{a^3}{3}\left(\frac{\pi}{2}-\frac{2}{3}\right).$$

Example 3. *Change into polar co-ordinates and evaluate*

$$\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,dy\,dx.$$

(P.T.U., Dec. 2003, May 2011, Dec. 2011)

Sol. For the region of integration in cartesian co-ordinates, y varies from 0 to ∞ and x also varies from 0 to ∞. Thus the region of integration is the plane XOY. Changing to polar co-ordinates by putting $x = r\cos\theta$, $y = r\sin\theta$ so that $x^2 + y^2 = r^2$; for the region of integration r varies from 0 to ∞ and θ varies from 0 to $\frac{\pi}{2}$.



$$\therefore \int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,dy\,dx = \int_0^{\pi/2}\int_0^{\infty} e^{-r^2}\cdot r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^{\infty} \frac{1}{2}\cdot e^{-r^2}\cdot 2r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^{\infty} \frac{1}{2} e^{-t} dt\,d\theta, \text{where } t = r^2$$

$$= \int_0^{\pi/2}\left[-\frac{1}{2}e^{-t}\right]_0^{\infty} d\theta = -\frac{1}{2}\int_0^{\pi/2}(0-1)\,d\theta = \frac{1}{2}\left[\theta\right]_0^{\pi/2} = \frac{\pi}{4}.$$

Example 4. *Evaluate the following by changing to polar coordinates*

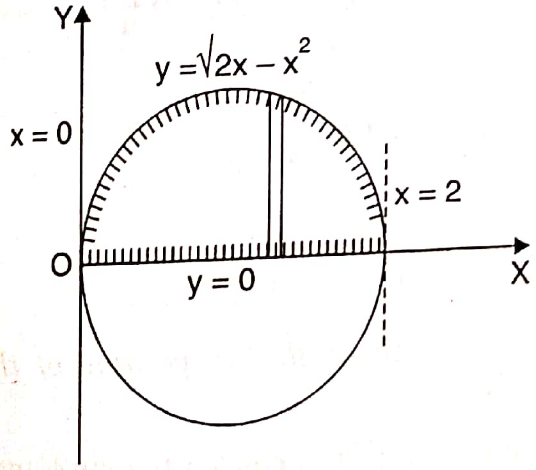
(*i*) $\int_0^2\int_0^{\sqrt{2x-x^2}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}}$ **(P.T.U., May 2012)**

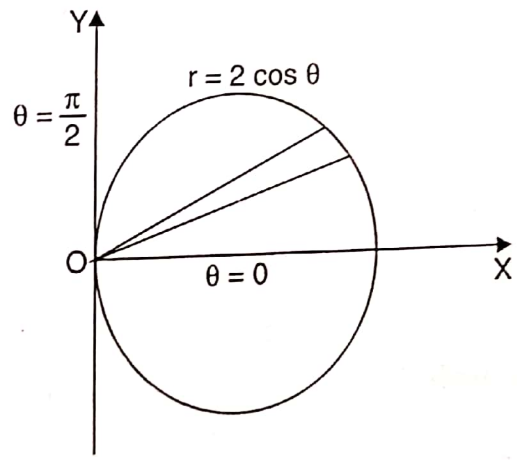
(*ii*) $\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \log_e(x^2+y^2+1)\,dx\,dy$ **(P.T.U., May 2003)**

Sol. (*i*) In the given integral, y varies from 0 to $\sqrt{2x-x^2}$ and x varies from 0 to 2.

$$y = \sqrt{2x-x^2} \Rightarrow y^2 = 2x - x^2 \Rightarrow x^2 + y^2 = 2x.$$

In polar co-ordinates, we have $r^2 = 2r\cos\theta$ or $r = 2\cos\theta$.





$\therefore$ For the region of integration, r varies from 0 to $2\cos\theta$ and θ varies from 0 to $\frac{\pi}{2}$.

In the given integral, replacing x by $r\cos\theta$, y by $r\sin\theta$, $dy\,dx$ by $r\,dr\,d\theta$, we have

$$I = \int_0^{\pi/2}\int_0^{2\cos\theta} \frac{r\cos\theta\cdot r\,dr\,d\theta}{r} = \int_0^{\pi/2}\int_0^{2\cos\theta} r\cos\theta\,dr\,d\theta$$

$$= \int_0^{\pi/2}\cos\theta\left[\frac{r^2}{2}\right]_0^{2\cos\theta} d\theta = \int_0^{\pi/2} 2\cos^3\theta\,d\theta = 2\cdot\frac{2}{3} = \frac{4}{3}.$$

(*ii*) $\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \log_e(x^2+y^2+1)\,dx\,dy$

In polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ $dxdy = r\,dr\,d\theta$

Here x varies from $-\sqrt{1-y^2}$ to $\sqrt{1-y^2}$

i.e., $x = \sqrt{1-y^2}$ or $x^2 + y^2 = 1$ or $r^2 = 1$ or $r = 1$ and y varies from -1 to 1

i.e., $r \sin\theta = -1$ and $r \sin\theta = 1$ put $r = 1$, we get $\sin\theta = -1$ and 1 $\therefore$ $\theta = -\frac{\pi}{2}$ and $\frac{\pi}{2}$

$\therefore$ In polar coordinates r varies from 0 to 1 and θ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

$$\therefore \quad \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \log_e (x^2 + y^2 + 1)\, dx\, dy = \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{r=0}^{1} \log_e (r^2 + 1) r\, dr\, d\theta$$

Put $r^2 + 1 = t$ $\therefore$ $2r\, dr = dt$ in the above integral

$$= \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{t=1}^{2} \log_e t \left(\frac{1}{2} dt\right) d\theta = \frac{1}{2}\int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{t=1}^{2} \log_e t\, dt\, d\theta$$

Integrate by parts

$$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[(\log t)\cdot t \Big|_1^2 - \int_1^2 \frac{1}{t}\, t\, dt \right] d\theta = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\log 2 - t \Big|_1^2 d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2\log 2 - 1)\, d\theta = \frac{1}{2}(2\log 2 - 1)\theta \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{1}{2}(2\log 2 - 1)\pi = \frac{\pi}{2}(2\log 2 - 1).$$

Example 5. *Evaluate* $\iiint z(x^2 + y^2 + z^2)\, dx\, dy\, dz$ *through the volume of the cylinder* $x^2 + y^2 = a^2$ *intercepted by the planes* $z = 0$ *and* $z = h$.

Sol. Changing to cylindrical co-ordinates by changing x to $r\cos\phi$, y to $r\sin\phi$ and replacing $dx\, dy\, dz$ by $r\, dr\, d\phi\, dz$

$$I = \int_0^h \int_0^{2\pi} \int_0^a z(r^2 + z^2)\, r\, dr\, d\phi\, dz = \int_0^h \int_0^{2\pi} \int_0^a (zr^3 + z^3 r)\, dr\, d\phi\, dz$$

$$= \int_0^h \int_0^{2\pi} \left[z\cdot\frac{r^4}{4} + z^3\cdot\frac{r^2}{2} \right]_0^a d\phi\, dz = \int_0^h \int_0^{2\pi} \left(\frac{a^4}{4} z + \frac{a^2}{2} z^3 \right) d\phi\, dz$$

$$= \int_0^h \left(\frac{a^4}{4} z + \frac{a^2}{2} z^3 \right) \Big[\phi\Big]_0^{2\pi} dz = \int_0^h 2\pi \left(\frac{a^4}{4} z + \frac{a^2}{2} z^3 \right) dz$$

$$= 2\pi \left[\frac{a^4 z^2}{8} + \frac{a^2 z^4}{8} \right]_0^h = \frac{\pi}{4}(a^4 h^2 + a^2 h^4) = \frac{\pi}{4} a^2 h^2 (a^2 + h^2).$$

Example 6. *Prove that* $\iiint_V \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}}\,dx\,dy\,dz = \frac{\pi^2}{4}\,abc,$

where
$$V = \left\{(x, y, z);\ \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \le 1\right\}.$$

Sol. Here $\quad V = \left\{(x,y,z);\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \le 1\right\}$ where V the volume of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Put $\frac{x}{a}=u,\ \frac{y}{b}=v,\ \frac{z}{c}=w \quad \therefore \quad x = au,\ y = bv,\ z = cw$ and $|\,J\,| = abc$

$$\therefore \quad I = \iiint_V \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}}\,dx\,dy\,dz = \iiint_{V'} \sqrt{1-u^2-v^2-w^2}\ abc\,du\,dv\,dw$$

where V′ is the volume of the sphere $u^2 + v^2 + w^2 = 1$.

Change it to spherical polar coordinates

$$u = r\sin\theta\cos\phi,\ v = r\sin\theta\sin\phi,\ z = r\cos\theta$$

and
$$du\,dv\,dw = r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$\therefore \quad I = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{1}\sqrt{1-r^2}\ abc\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= abc\int_0^{2\pi}\int_0^{\pi}\left[\int_0^1 r^2\sqrt{1-r^2}\,dr\right]\sin\theta\,d\theta\,d\phi \qquad ...(1)$$

To integrate $\int_0^1 r^2\sqrt{1-r^2}\,dr$; put $r = \sin t \quad \therefore \quad dr = \cos t\,dt$

$$\therefore \quad \int_0^1 r^2\sqrt{1-r^2}\,dr = \int_0^{\frac{\pi}{2}} \sin^2 t\cos^2 t\,dt = \frac{1.1}{4.2}\frac{\pi}{2}=\frac{\pi}{16}$$

From (1)
$$I = abc\int_0^{2\pi}\int_0^{\pi}\frac{\pi}{16}\sin\theta\,d\theta\,d\phi = \frac{abc\pi}{16}\int_0^{2\pi} -\cos\theta\Big|_0^{\pi}\,d\phi$$

$$= \frac{abc\pi}{16}\cdot 2\int_0^{2\pi}d\phi = \frac{abc\pi}{8}\cdot\phi\Big|_0^{2\pi} = \frac{abc\pi}{8}\,2\pi = \frac{abc\,\pi^2}{4}.$$

Example 7. *Evaluate* $\iiint \frac{dx dy dz}{\sqrt{1-x^2-y^2-z^2}}$, *the integral being extended to the positive octant of the sphere* $x^2 + y^2 + z^2 = 1$. **(P.T.U., May 2004, May 2008)**

Sol. In the positive octant of the sphere

$$0 \le z \le \sqrt{1-x^2-y^2}\ ;\ 0 \le y \le \sqrt{1-x^2}\ ;\ 0 \le x \le 1$$

$\therefore \quad \int_0^1\int_0^{\sqrt{1-y^2}}\int_0^{\sqrt{1-x^2-y^2}}\frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}}$ represents the volume of the sphere in positive octant.

Changing to spherical polar co-ordinates by putting $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ so that $\quad x^2 + y^2 + z^2 = r^2.$

For the volume of sphere $x^2 + y^2 + z^2 = 1$ in the positive octant, r varies from 0 to 1, θ varies from 0 to $\frac{\pi}{2}$ and ϕ varies from 0 to $\frac{\pi}{2}$.

Replacing $dz\,dy\,dx$ by $r^2 \sin\theta\, dr\, d\theta\, d\phi$, we have

$$I = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \frac{r^2 \sin\theta\, dr\, d\theta\, d\phi}{\sqrt{1-r^2}} = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \frac{1-(1-r^2)}{\sqrt{1-r^2}} \sin\theta\, dr\, d\theta\, d\phi$$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \left(\frac{1}{\sqrt{1-r^2}} - \sqrt{1-r^2}\right) \sin\theta\, dr\, d\theta\, d\phi$$

$$= \int_0^{\pi/2}\int_0^{\pi/2} \sin\theta \left[\sin^{-1} r - \left(\frac{r\sqrt{1-r^2}}{2} + \frac{1}{2}\sin^{-1} r\right)\right]_0^1 d\theta\, d\phi$$

$$= \int_0^{\pi/2}\int_0^{\pi/2} \sin\theta \left[\frac{\pi}{2} - \frac{1}{2}\cdot\frac{\pi}{2}\right] d\theta\, d\phi = \int_0^{\pi/2} \frac{\pi}{4}\Big[-\cos\theta\Big]_0^{\pi/2} d\phi = \int_0^{\pi/2} \frac{\pi}{4}\, d\phi$$

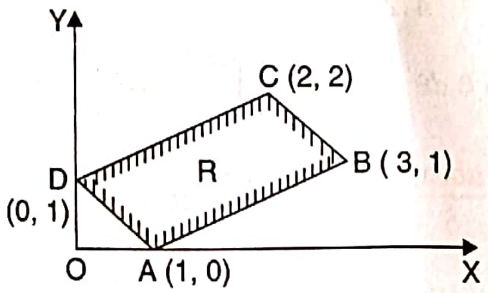
$$= \frac{\pi}{4}\cdot\frac{\pi}{2} = \frac{\pi^2}{8}.$$

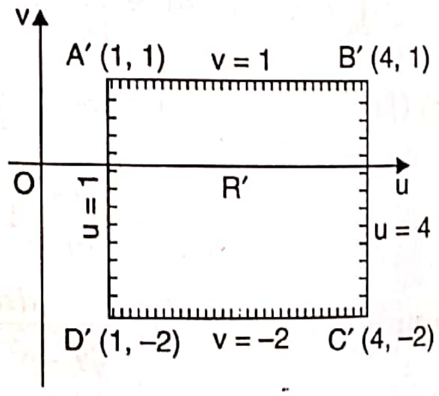
Note. For the whole volume of the sphere $x^2 + y^2 + z^2 = a^2$.

$$0 \le r \le a,\ 0 \le \theta \le \pi,\ 0 \le \phi \le 2\pi.$$

Example 8. *Evaluate* $\iint_R (x+y)^2\, dx\, dy$, *where R is the parallelogram in the xy-plane with vertices (1, 0), (3, 1), (2, 2), (0, 1), using the transformation* $u = x + y$ *and* $v = x - 2y$.

Sol. The vertices A(1, 0), B(3, 1), C(2, 2), D(0, 1) of the parallelogram ABCD in the xy-plane become A′(1, 1), B′(4, 1), C′(4, –2), D′(1, –2) in the uv-plane under the given transformation.





The region R in the xy-plane becomes the region R′ in the uv-plane which is a rectangle bounded by the line $u = 1$, $u = 4$ and $v = -2$, $v = 1$. Solving the given equations for x and y, we have $x = \frac{1}{3}(2u + v)$, $y = \frac{1}{3}(u - v)$

$$\therefore \qquad J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{vmatrix} = -\frac{1}{3}$$

$$\therefore \quad \iint_R (x+y)^2\, dx\, dy = \iint_{R'} u^2\, |J|\, du\, dv = \int_{-2}^{1} \int_{1}^{4} u^2 \cdot \frac{1}{3}\, du\, dv$$

$$= \int_{-2}^{1} \frac{1}{3}\left[\frac{u^3}{3}\right]_1^4 dv = \int_{-2}^{1} 7\, dv = 7\Big[v\Big]_{-2}^{1} = 7 \times 3 = 21.$$

Example 9. (*i*) *Using the transformation* $x + y = u$, $y = uv$ *show that*

$\iint \sqrt{xy(1-x-y)}\; dx\, dy = \dfrac{2\pi}{105}$; *integration being taken over the area of the triangle bounded by the lines* $x = 0$, $y = 0$, $x + y = 1$. **(P.T.U., Dec. 2004)**

(*ii*) *Evaluate* $\int_0^1 \int_0^{1-x} e^{\frac{y}{x+y}}\, dy\, dx$ *by using the transformation* $x + y = u$, $y = uv$.

Sol. (*i*) Given relation is $x + y = u$, $y = uv$

transforms x, y to u, v we get

$$x + y = u,\ y = uv$$

$$\Rightarrow \qquad x = u - uv = u(1-v)$$

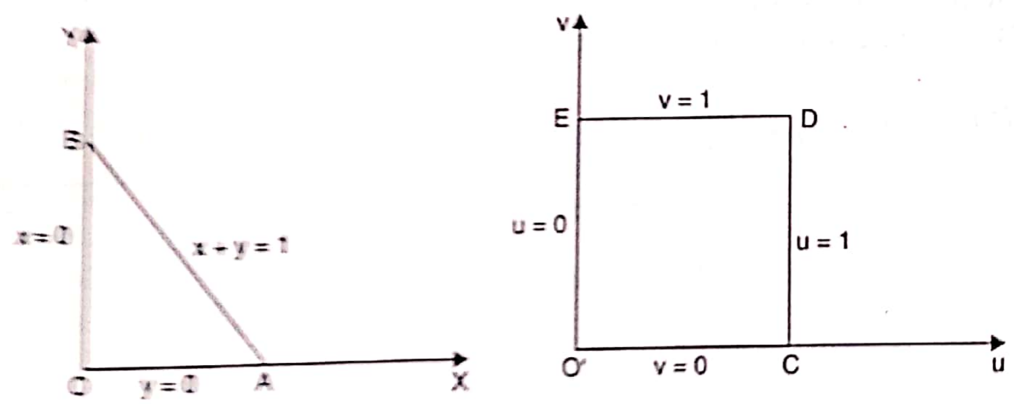
i.e.,
$$\left.\begin{aligned} x &= u\,(1-v) \\ y &= uv \end{aligned}\right\}$$

Now $x = 0 \;\Rightarrow\; u = 0$ or $v = 1$

$y = 0 \;\Rightarrow\; u = 0$ or $v = 0$

$x + y = 1 \;\Rightarrow\; u - uv + uv = 1 \;\Rightarrow\; u = 1$

$\therefore$ Triangle transforms into a square as shown in the figure.



We know that

$$\iint_R f(x, y)\, dx\, dy = \iint_{R'} f[\phi(u, v), \psi(u, v)]\, |J|\, du\, dv$$

$$\therefore \qquad |J| = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2mm] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u - uv + uv = u$$

$$\therefore \quad \iint [xy(1-x-y)]^{1/2}\, dx\, dy = \int_{u=0}^{1} \int_{v=0}^{1} [u(1-v)\, uv(1-u)]^{1/2}\, u\, du\, dv$$

$$= \int_0^1 \left(u^2 \sqrt{1-u}\right) du \int_0^1 \sqrt{v - v^2}\; dv \qquad \ldots(1)$$

Put $u = \sin^2 \theta$

$du = 2 \sin \theta \cos \theta \, d\theta$

(1) becomes

$$= \int_0^{\pi/2} \sin^4 \theta \cdot \cos \theta \, 2 \sin \theta \cos \theta \, d\theta \int_0^1 \sqrt{\left(\frac{1}{2}\right)^2 - \left(v - \frac{1}{2}\right)^2} \, dv$$

$$= \int_0^{\pi/2} 2 \sin^5 \theta \cdot \cos^2 \theta \, d\theta \left[\frac{\left(v - \frac{1}{2}\right)\sqrt{\left(\frac{1}{2}\right)^2 - \left(v - \frac{1}{2}\right)^2}}{2} + \frac{\frac{1}{4}}{2} \sin^{-1} \frac{v - \frac{1}{2}}{1/2} \right]_0^1$$

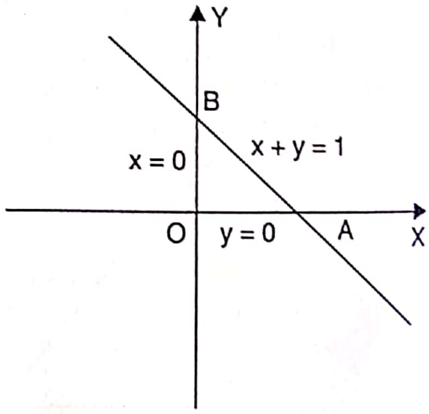
$$= 2 \, \frac{4 \cdot 2 \cdot 1}{7 \cdot 5 \cdot 3 \cdot 1} \left[0 + \frac{1}{8} \frac{\pi}{2} - \frac{1}{8}\left(-\frac{\pi}{2}\right) \right]$$

$$= \frac{16}{105} \times \frac{\pi}{8} = \frac{2\pi}{105}.$$

(ii) $\int_0^1 \int_0^{1-x} e^{\frac{y}{x+y}} \, dy \, dx$

Limits of integration of y are from 0 to $1 - x$, *i.e.*, $(y = 0)$ $y = 1 - x$ or $x + y = 1$ and limits of integration of x are from 0 to 1. In this case also the region of integration is same as in part (*i*)

$\therefore$ Under the transformation $x + y = u$, $y = uv$ the triangle OAB transforms to the square O′CDE



$$\therefore \int_0^1 \int_0^{1-x} e^{\frac{y}{x+y}} \, dy \, dx = \int_0^1 \int_0^1 e^{\frac{uv}{u}} \, |J| \, du \, dv$$

$| J | = u$ proved in (*i*) part

$$= \int_0^1 \int_0^1 e^v \, u \, du \, dv$$

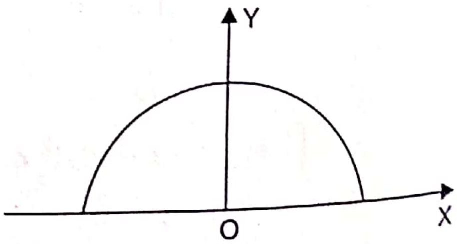
$$= \int_0^1 \int e^v \left[\frac{u^2}{2} \Big|_0^1 \right] dv$$

$$= \int_0^1 e^v \cdot \frac{1}{2} dv = \frac{1}{2} e^v \Big|_0^1 = \frac{e^1 - e^0}{2} = \frac{e - 1}{2}.$$

Example 10. *Transform* $\int_0^{\pi} \int_0^{a} r^3 \sin \theta \cos \theta \, dr \, d\theta$ *to cartesian form and evaluate.*

Sol. In the given integral r varies from 0 to a and θ varies from 0 to π ; $r = a$ represents a circle

$\therefore$ given region of integration is semicircle in the upper half plane the relation between polar and cartesian system is



$$x = r \cos \theta, \; y = r \sin \theta$$

[**Hint.** See S.E. 5]

10. Evaluate the following integrals through the volume of the sphere $x^2 + y^2 + z^2 = 1$, by changing into spherical polar co-ordinates:

(*i*) $\iiint z^2 \, dx \, dy \, dz$ (*ii*) $\iiint (x^2 + y^2 + z^2)^m \, dx \, dy \, dz$. $(m > 0)$

[**Hint.** See S.E. 7]

Answers

1. 2 2. $(e-1)\frac{\pi}{2}$ 3. $\frac{1}{14}$

4. (*i*) $\frac{\pi a}{4}$ (*ii*) πa^2 (*iii*) $8\left(\frac{\pi}{2} - \frac{5}{3}\right)a^2$ (*iv*) $\frac{\pi a^4}{8}$

(*v*) $\frac{\pi a^5}{20}$ (*vi*) $\frac{a^3}{3} \log(\sqrt{2} + 1)$ 5. $\pi(1 - e^{-a^2})$ 6. $\frac{2^{n+3}}{n+4}$

7. $\frac{\pi ab}{8}(\pi - 2)$ 8. $\frac{\pi^2}{8}$ 9. $\frac{5\pi}{4}$

10. (*i*) $\frac{4\pi}{15}$ (*ii*) $\frac{4\pi}{2m+3}$.

6.7. AREA BY DOUBLE INTEGRATION

(*a*) *Cartesian Co-ordinates.* The area A of the region bounded by the curves $y = f_1(x)$, $y = f_2(x)$ and the lines $x = a$, $x = b$ is given by $A = \int_a^b \int_{f_1(x)}^{f_2(x)} dy \, dx$

The area A of the region bounded by the curves $x = f_1(y)$, $x = f_2(y)$ and the lines $y = c$, $y = d$ is given by

$$A = \int_c^d \int_{f_1(y)}^{f_2(y)} dx \, dy.$$

(*b*) *Polar Co-ordinates.* The area A of the region bounded by the curves $r = f_1(\theta)$, $r = f_2(\theta)$ and the lines $\theta = \alpha$, $\theta = \beta$ is given by $A = \int_\alpha^\beta \int_{f_1(\theta)}^{f_2(\theta)} r \, dr \, d\theta$

6.8. VOLUME BY DOUBLE INTEGRATION

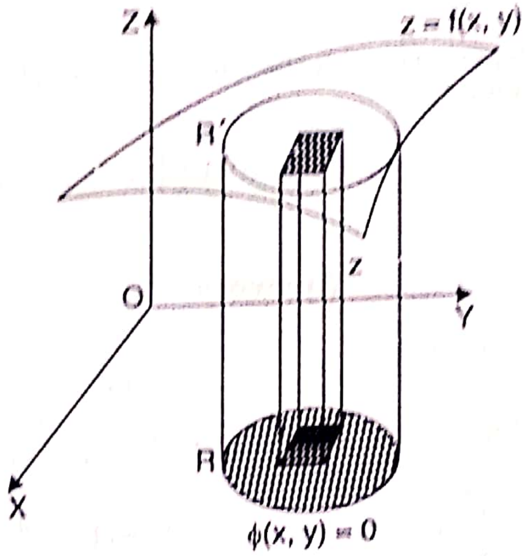
(*a*) *Cartesian Co-ordinates.* Consider a surface $z = f(x, y)$ ...(1)

Let the orthogonal projection of its region R′ on the *xy*-plane be the region R given by $\phi(x, y) = 0$. ...(2)

Now (2) represents a cylinder with generators parallel to *z*-axis and the guiding curve given by (2). Let V be the volume of this cylinder between R and R′.

Divide R into elementary rectangles of area $\delta x \, \delta y$ by drawing lines parallel to the axis of *x* and *y*. On each of these rectangles, erect prisms of lengths parallel to *z*-axis. Volume of this prism between R and R′ is $z \, \delta x \, \delta y$. The volume V is composed of such prisms.

$$\therefore \quad V = \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} \sum z\, \delta x\, \delta y = \iint_R z\, dx\, dy.$$



(*b*) *Cylindrical Co-ordinates.* Let the equation of the surface be $z = f(r, \phi)$. Replacing $dx\, dy$ by $r\, dr\, d\phi$, we get $V = \iint_R zr\, dr\, d\phi$.

ILLUSTRATIVE EXAMPLES

Example 1. (*a*) *Find the area lying between the parabola $y = 4x - x^2$ and the line $y = x$.*
(P.T.U., May 2003)

(*b*) *Find by double integration the area bounded by $x = 2y - y^2$ and $x = y^2$.*
(P.T.U., Dec., 2011)

(*c*) *Find the area between the parabolas $y^2 = 4ax$, $x^2 = 4ay$.*
(P.T.U., Dec. 2003, Dec. 2012, May 2015)

Sol. (*a*) The two curves intersect at points whose abscissae are given by $4x - x^2 = x$

or $\quad x^2 - 3x = 0 \quad$ *i.e.,* $\quad x = 0, 3.$

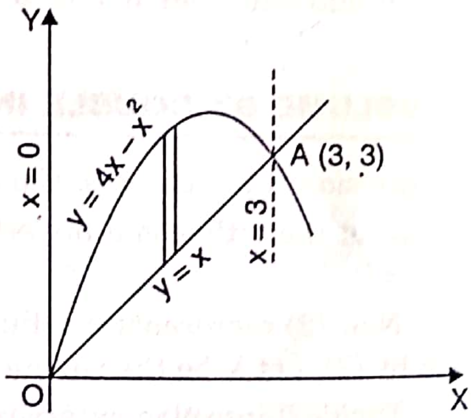
Using vertical strips, the required area lies between $x = 0$, $x = 3$ and $y = x$, $y = 4x - x^2$.

$$\therefore \quad \text{Required area} = \int_0^3 \int_x^{4x - x^2} dy\, dx$$

$$= \int_0^3 \left[y \right]_x^{4x - x^2} dx$$

$$= \int_0^3 (4x - x^2 - x)\, dx = \int_0^3 (3x - x^2)\, dx$$

$$= \left[\frac{3x^2}{2} - \frac{x^3}{3} \right]_0^3 = \frac{27}{2} - 9 = 4.5.$$



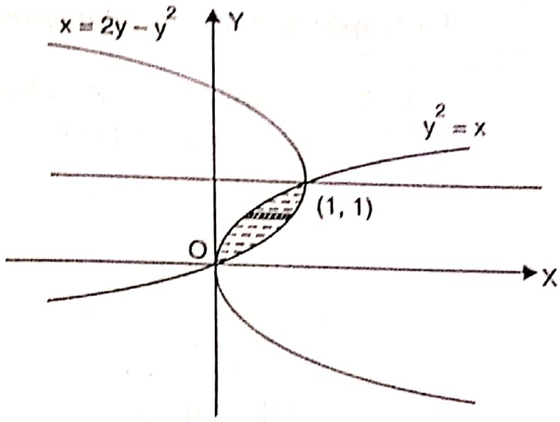
(*b*) Equations of the curves are $x = 2y - y^2$ and $x = y^2$. Both are parabolas.

They intersect at points given by
$y^2 = 2y - y^2$ or $2y^2 - 2y = 0$ or $2y(y-1) = 0$
or $y = 0, y = 1$ or at (0, 0) and (1, 1)

Equation of the parabola $x = 2y - y^2$ can be put in the form $x = -(y^2 - 2y)$

$x - 1 = -(y^2 - 2y + 1) = -(y-1)^2$

or $(y-1)^2 = -(x-1)$, which is a parabola having vertex at (1, 1)



Using horizontal strip

$$\text{Required area} = \int_0^1 \int_{y^2}^{2y-y^2} dx\,dy$$

$$= \int_0^1 x \Big|_{y^2}^{2y-y^2} dy = \int_0^1 (2y - y^2 - y^2)\,dy = \int_0^1 (2y - 2y^2)\,dy$$

$$= 2\left[\frac{y^2}{2} - \frac{y^3}{3}\Big|_0^1\right] = 2\cdot\left[\frac{1}{2} - \frac{1}{3}\right] = 2\cdot\frac{1}{6} = \frac{1}{3}.$$

(c) The two curves $y^2 = 4ax$

and $x^2 = 4ay$

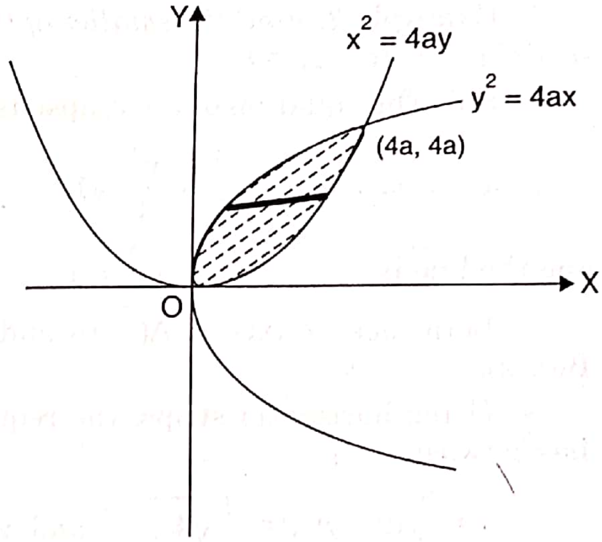
intersect at $\dfrac{x^4}{16a^2} = 4ax$

or $x^4 - 64a^3x = 0$

or $x(x^3 - 64a^3) = 0$

i.e., at $x = 0$ and $x = 4a$

i.e., at (0, 0) and (4*a*, 4*a*)



Using horizontal strips; x varies from $\dfrac{y^2}{4a}$ to $2\sqrt{ay}$ and y varies from 0 to $4a$.

$$\therefore \text{ Required area} = \int_0^{4a} \int_{\frac{y^2}{4a}}^{2\sqrt{ay}} dx\,dy = \int_0^{4a} x \Big|_{\frac{y^2}{4a}}^{2\sqrt{ay}} dy$$

$$= \int_0^{4a} \left(2\sqrt{ay} - \frac{y^2}{4a}\right) dy$$

$$= 2\sqrt{a}\,\frac{y^{3/2}}{3/2} - \frac{y^3}{12a}\Big|_0^{4a}$$

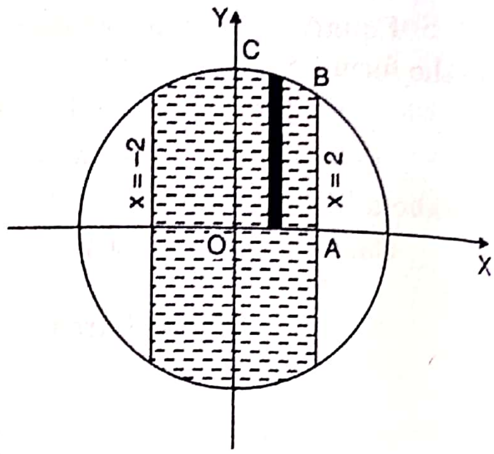
$$= \frac{4\sqrt{a}}{3}(4a)^{3/2} - \frac{(4a)^3}{12a} = \frac{4}{3}\sqrt{a}\cdot 8a^{3/2} - \frac{64a^3}{12a}$$

$$= \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2.$$

Example 2. *Find the area of the region bounded by the lines $x = -2$, $x = 2$ and the circle $x^2 + y^2 = 9$.* **(P.T.U., May 2007)**

Sol. The equation on of the circles $x^2 + y^2 = 9$

Required area = 4 OABC



$$= 4 \int_0^2 \int_0^{\sqrt{9-x^2}} dy\, dx$$

$$= 4 \int_0^2 y \Big|_0^{\sqrt{9-x^2}} dx$$

$$= 4 \int_0^2 \sqrt{9-x^2}\, dx$$

$$= 4 . \left\{ \frac{x\sqrt{9-x^2}}{2} + \frac{9}{2} \sin^{-1} \frac{x}{3} \Big|_0^2 \right\}$$

$$= 4 \left\{ \frac{2\sqrt{5}}{2} + \frac{9}{2} \sin^{-1} \frac{2}{3} \right\}$$

$$= 4\sqrt{5} + 18 \sin^{-1} 2/3.$$

Example 3. *Find the smaller of the areas bounded by the ellipse $4x^2 + 9y^2 = 36$ and the straight line $2x + 3y = 6$.*

Sol. The equation of the ellipse is

$$\frac{x^2}{9} + \frac{y^2}{4} = 1 \qquad \text{...(1)}$$

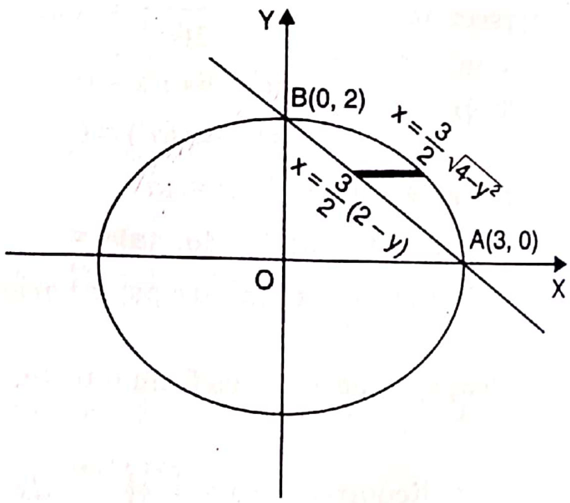
and the line is

$$\frac{x}{3} + \frac{y}{2} = 1 \qquad \text{...(2)}$$

Both meet x-axis at A(3, 0) and y-axis at B(0, 2).

Using horizontal strips, the required area lies between

$$x = \frac{3}{2}(2-y),\ x = \frac{3}{2}\sqrt{4-y^2} \text{ and } y = 0, y = 2.$$



$$\therefore \quad \text{Required area} = \int_0^2 \int_{\frac{3}{2}(2-y)}^{\frac{3}{2}\sqrt{4-y^2}} dx\, dy$$

$$= \int_0^2 \Big[x \Big]_{\frac{3}{2}(2-y)}^{\frac{3}{2}\sqrt{4-y^2}} dy$$

$$= \int_0^2 \frac{3}{2} [\sqrt{4-y^2} - (2-y)]\, dy$$

$$= \frac{3}{2} \left[\frac{y\sqrt{4-y^2}}{2} + \frac{4}{2} \sin^{-1} \frac{y}{2} - 2y + \frac{y^2}{2} \right]_0^2$$

$$= \frac{3}{2} [2 \sin^{-1} 1 - 4 + 2] = \frac{3}{2}\left(2 \cdot \frac{\pi}{2} - 2\right) = \frac{3}{2}(\pi - 2).$$

6.9. VOLUME BY TRIPLE INTEGRATION

The volume V of a three dimensional region is given by $V = \iiint_V dx\, dy\, dz$.

If the region is bounded by $x = f_1(y, z)$, $x = f_2(y, z)$; $y = \phi_1(z)$, $y = \phi_2(z)$ and $z = a$, $z = b$, then

$$V = \int_a^b \int_{\phi_1(z)}^{\phi_2(z)} \int_{f_1(y,z)}^{f_2(y,z)} dx\, dy\, dz$$

The order of integration may be changed with a suitable change in the limits of integration.

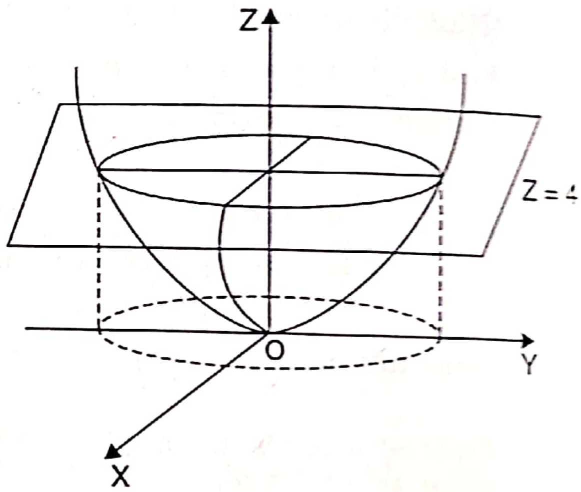
In cylindrical co-ordinates, we have $V = \iiint_V r\, dr\, d\phi\, dz$

In spherical polar co-ordinates, we have $V = \iiint_V r^2 \sin\theta\, dr\, d\theta\, d\phi$.

ILLUSTRATIVE EXAMPLES

Example 1. *Find by triple integration, the volume of the paraboloid of revolution $x^2 + y^2 = 4z$ cut off by the plane $z = 4$.*

Sol. By symmetry, the required volume is 4 times the volume in the positive octant. The volume in the positive octant is bounded on the sides by the zx and yz-planes ; from above by the plane $z = 4$ and below by the curved surface $x^2 + y^2 = 4z$.



The section of the paraboloid by the plane $z = 4$ is the circle $x^2 + y^2 = 16$, $z = 4$ and its projection on the xy-plane is the circle $x^2 + y^2 = 16$, $z = 0$.

The volume in the positive octant is bounded by $z = \dfrac{x^2 + y^2}{4}$, $z = 4$, $y = 0$, $y = \sqrt{16 - x^2}$ and $x = 0$, $x = 4$.

∴ Required volume

$$= 4 \int_0^4 \int_0^{\sqrt{16-x^2}} \int_{\frac{(x^2+y^2)}{4}}^{4} dz\, dy\, dx = 4 \int_0^4 \int_0^{\sqrt{16-x^2}} \Big[z \Big]_{\frac{x^2+y^2}{4}}^{4} dy\, dx$$

$$= 4 \int_0^4 \int_0^{\sqrt{16-x^2}} \left(4 - \frac{x^2 + y^2}{4}\right) dy\, dx = 4 \int_0^4 \left[\left(4 - \frac{x^2}{4}\right) y - \frac{1}{4} \cdot \frac{y^3}{3}\right]_0^{\sqrt{16-x^2}} dx$$

$$= 4 \int_0^4 \left[\left(4 - \frac{x^2}{4}\right)\sqrt{16 - x^2} - \frac{1}{12}(16 - x^2)^{3/2}\right] dx$$

$$= 4\int_0^4 \left[\frac{1}{4}(16-x^2)\sqrt{16-x^2} - \frac{1}{12}(16-x^2)^{3/2}\right] dx$$

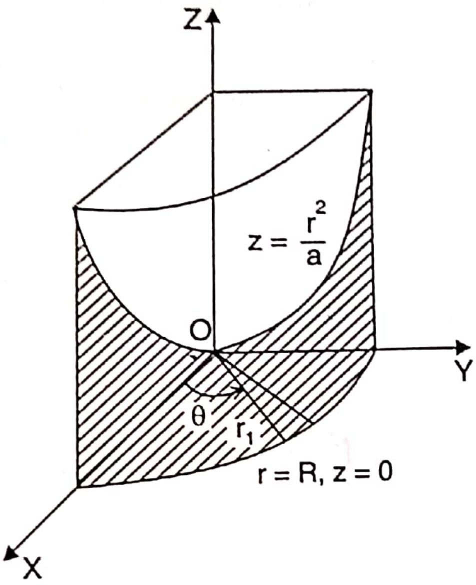
$$= 4\int_0^4 \frac{1}{6}(16-x^2)^{3/2}\, dx = \frac{2}{3}\int_0^4 (16-x^2)^{3/2}\, dx$$

$$= \frac{2}{3}\int_0^{\pi/2} (16)^{3/2} \cdot \cos^3\theta \cdot 4\cos\theta\, d\theta, \text{ where } x = 4\sin\theta$$

$$= \frac{512}{3}\int_0^{\pi/2} \cos^4\theta\, d\theta = \frac{512}{3}\cdot\frac{3\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2} = 32\pi.$$

Example 2. *Find, by triple integration, the volume of the region bounded by the paraboloid* $az = x^2 + y^2$ *and the cylinder* $x^2 + y^2 = R^2$.

Sol. Changing to cylindrical co-ordinates, by putting $x = r\cos\theta$, $y = r\sin\theta$ the equation of the paraboloid becomes $az = r^2$ or $z = \frac{r^2}{a}$ and the equation of the cylinder becomes $r^2 = R^2$ or $r = R$. On account of symmetry, the required volume is four times the volume in the positive octant. Thus, in the common region, z varies from 0 to $\frac{r^2}{a}$, r varies from 0 to R and θ varies from 0 to $\frac{\pi}{2}$.



$$\therefore \text{ Required volume} = 4\int_0^{\pi/2}\int_0^R\int_0^{r^2/a} r\, dz\, dr\, d\theta$$

$$= 4\int_0^{\pi/2}\int_0^R r\Big[z\Big]_0^{r^2/a} dr\, d\theta$$

$$= 4\int_0^{\pi/2}\int_0^R r\cdot\frac{r^2}{a}\, dr\, d\theta = 4\int_0^{\pi/2}\left[\frac{r^4}{4a}\right]_0^R d\theta$$

$$= \frac{1}{a}\int_0^{\pi/2} R^4\, d\theta = \frac{R^4}{a}\cdot\frac{\pi}{2} = \frac{\pi R^4}{2a}.$$

Example 3. *Find, by triple integration, the volume of a sphere of radius a.*

(P.T.U., May 2003, May 2015)

Sol. Changing to spherical polar co-ordinates by putting $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ so that $x^2 + y^2 + z^2 = r^2$.

The equation of a sphere of radius a in cartesian co-ordinates is $x^2 + y^2 + z^2 = a^2$

The same equation in spherical polar co-ordinates is $r^2 = a^2$ or $r = a$.

On account of symmetry, the required volume is 8 times the volume of the sphere in the positive octant for which r varies from 0 to a, θ varies from 0 to $\frac{\pi}{2}$ and ϕ varies from 0 to $\frac{\pi}{2}$.

$$\therefore \text{ Required volume} = 8\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$= 8\int_0^{\pi/2}\int_0^{\pi/2} \sin\theta \cdot \left[\frac{r^3}{3}\right]_0^a d\theta\, d\phi = 8\int_0^{\pi/2}\int_0^{\pi/2} \frac{a^3}{3} \sin\theta\, d\theta\, d\phi$$

$$= 8\int_0^{\pi/2} -\frac{a^3}{3}\Big[\cos\theta\Big]_0^{\pi/2} d\phi = \frac{8}{3}a^3\int_0^{\pi/2} d\phi = \frac{8}{3}a^3 \cdot \frac{\pi}{2} = \frac{4}{3}\pi a^3.$$

Example 4. *Find the volume of the region enclosed by the surfaces* $z = x^2 + 3y^2$, $z = 8 - x^2 - y^2$. **(P.T.U., May 2003)**

Sol. Equations of the surfaces are

$$z = x^2 + 3y^2,\ z = 8 - x^2 - y^2$$

Their intersections are given by

$$x^2 + 3y^2 = 8 - x^2 - y^2 \quad \text{or} \quad 2x^2 + 4y^2 = 8 \quad \text{or} \quad x^2 + 2y^2 = 4$$

$\therefore$ z varies from $x^2 + 3y^2$ to $8 - x^2 - y^2$

y varies from $-\sqrt{\dfrac{4-x^2}{2}}$ to $\sqrt{\dfrac{4-x^2}{2}}$

and x varies from -2 to 2

$$\text{Volume of the Region} = \int_{-2}^{2}\int_{-\sqrt{\frac{4+x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}}\int_{x^2+3y^2}^{8-x^2-y^2} dz\, dy\, dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} (8 - x^2 - y^2 - x^2 - 3y^2)\, dy\, dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} (8 - 2x^2 - 4y^2)\, dy\, dx$$

$$= \int_{-2}^{2} 8y - 2x^2y - \frac{4y^3}{3}\Bigg|_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} dx = \int_{-2}^{2} (8-2x^2)\,2\sqrt{\frac{4-x^2}{2}} - \frac{4}{3}\cdot 2\left(\frac{4-x^2}{2}\right)^{3/2} dx$$

$$= \int_{-2}^{2} \frac{8}{3\sqrt{2}}(4-x^2)^{3/2}\, dx \quad \text{Put } x = 2\sin\theta;\ dx = 2\cos\theta\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{8}{3\sqrt{2}}(4 - 4\sin^2\theta)^{3/2} \cdot 2\cos\theta\, d\theta$$

$$= \frac{128}{3\sqrt{2}}\int_{-\pi/2}^{\pi/2} \cos^4\theta\, d\theta = \frac{128}{3\sqrt{2}}2\int_0^{\frac{\pi}{2}} \cos^4\theta\, d\theta = \frac{256}{3\sqrt{2}}\,\frac{3\cdot 1}{4\cdot 2}\,\frac{\pi}{2} = \frac{16}{\sqrt{2}}\pi = 8\sqrt{2}\,\pi.$$